On q-differential graded algebras and N-complexes

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Abstract We generalize a result of V. Abramov on q-differential graded algebras and show in explicit terms its relation to N-complexes.

1 Introduction

We begin by summarizing Abramov's result and ours for easy comparison.

1.1 Abramov's main result

In Abramov's setting (see [1]) we have a $\mathbb{Z}$-graded associative $\mathbb{C}$-algebra $D = \bigoplus_{n\in\mathbb{Z}} D_n$ with unity. Fundamental to his paper is the (graded) q-commutator, $q \in \mathbb{C}$, defined by

$$\langle a,b\rangle_q := ab - q^{\deg(a)\deg(b)}ba, \quad \text{for} \quad a \in D_{\deg(a)}, \quad \text{and} \quad b \in D_{\deg(b)},$$

and where $\deg(\cdot)$ is the graded degree-function. Notice that this is undefined for non-homogenous elements and that this definition uses more than the fact that $\mathbb{Z}$ is a group: it uses the fact that $\mathbb{Z}$ is a ring!

It is easy to see that

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$$\langle a,bc\rangle_q = \langle a,b\rangle_q c + q^{\deg(a)\deg(b)} b\langle a,c\rangle_q,$$

that is, the mapping $\mathrm{d}_a(b) := \langle x,\cdot\rangle_q(b) = \langle a,b\rangle_q$ is a q-differential on D. There is however one thing that should be stressed: d_a is only linear on homogeneous components! This is due to the involvement of the factor $q^{\deg(a)\deg(b)}$ and the fact that $\deg(\cdot)$ is not linear.

Abramov's main result can now be formulated as

Theorem 1 (Abramov [1]). *Suppose $D = \bigoplus_{n\in\mathbb{Z}} D_n$ and that $N \geq 2$ is given such that q is a primitive N^{th}-root of unity. Assume further that $a \in D_1$ and that $a^N = u\mathbf{1}_D \in D_0$, for $u \in \mathbb{C}$. Then $\mathrm{d}_a^N(b) = 0$ for all $b \in A$.*

1.2 Our main result

Let k be a commutative, associative ring with unity and A an associative k-algebra with unity. Furthermore, let G be a subset of A and form $k[\mathsf{G}]$, the k-algebra generated by G. Take a multiplicative map σ with domain G, and if not already linear, extend it k-linearly on $k[\mathsf{G}]$ by $\sigma(rg + r'g') := r\sigma(g) + r'\sigma(g')$. We assume that $\sigma(g) = \phi(a,g)g$ for a fixed $a \in A$ and a map $\phi : \{a\} \times \mathsf{G} \to \mathrm{Z}(A)$, where $\mathrm{Z}(A)$ is the center of A.

Put $\Delta(b) := [a,\cdot\rangle(b) = [a,b\rangle = ab - \sigma(b)a$, for $b \in k[\mathsf{G}]$. This is a σ-derivation on $k[\mathsf{G}]$. Compare this with Abramov's q-differential $\mathrm{d}_a = \langle a,\cdot\rangle_q$. Assume also that $a^N \in \mathrm{Z}(A)$ for some $N \geq 2$.

Theorem 2. *If $a \in k[\mathsf{G}]$ and $\phi(a,a)$ is a primitive N^{th}-root of unity and $\phi(a,b)^N = \mathbf{1}$ for all $b \in k[\mathsf{G}]$, then $\Delta^N(b) = 0$ for all $b \in k[\mathsf{G}]$.*

1.3 Comparison

First notice that if G is a generating set of A over k then $k[\mathsf{G}] = A$. This is also true if A is $\mathbb{Z}$-graded (for instance) and $\mathsf{G} = \cup_{n\in\mathbb{Z}} A_n$, the set of homogenous elements of A, since any $a \in A$ is a finite sum of homogenous elements.

In our approach we avoid the grading but we retain Abramov's result in the graded case. To see this assume that our algebra A is $\mathbb{Z}$-graded and that $k = \mathbb{C}$. By the above argument $A = k[\mathsf{G}]$ where G is the subset of A of homogeneous elements. For example, the map σ can now be defined as $\sigma(g) = \phi(a,g)g = q^{\deg(a)\deg(g)}g = q^{\deg(g)}g$ where we have $a \in A_1$ and $\phi(a,g) = q^{\deg(a)\deg(g)}$. Obviously this map is only linear on each homogeneous component and so has to be explicitly extended. From this we see that $\phi(a,a) = q$ and $\phi(a,b) = q^{\deg(b)}$, for $b \in \mathsf{G}$. Assuming further that $a^N = u\mathbf{1}$ we are exactly in Abramov's case.

2 Set-up

Let A be a k-algebra and N an A-bimodule. A *module derivation on A* is a k-linear map $\mathscr{D}: A \to N$ satisfying $\mathscr{D}(ab) = \mathscr{D}(a)b + a\mathscr{D}(b)$ for $a, b \in A$. Furthermore, let Γ and M be left A-modules (in particular k-modules). Then Γ is said to act on M if there is a k-linear map $\mu : \Gamma \otimes_k M \to M$. We write $\gamma.x$ for $\mu(\gamma \otimes_k x)$. A *general derivation* on (A, Γ, M) is a quadruple $(\sigma, \tau, \Delta, \mathscr{D})$ [4] where

- $\sigma, \Delta : \Gamma \to \Gamma$, and
- $\tau, \mathscr{D} : M \to M$

are all k-linear maps such that

$$\mathscr{D}(\gamma.x) = \Delta(\gamma).\tau(x) + \sigma(\gamma).\mathscr{D}(x). \tag{1}$$

Definition 1. If $\Gamma = M = A$ and $\mathscr{D} = \Delta$, then a general derivation $(\sigma, \tau, \Delta, \mathscr{D})$ is said to be a (σ, τ)*-derivation* on A and when $\tau = \mathrm{id}_M$ it is usually called a σ*-derivation*. Here we simply write this as Δ.

Assume that A is a k-algebra equipped with a k-endomorphism σ. Define the operator $[a, \cdot\rangle : A \to A$, for each $a \in A$, by:

$$\Delta(b) := [a, \cdot\rangle(b) := ab - \sigma(b)a, \tag{2}$$

i.e., $\Delta := [a, \cdot\rangle$. Clearly Δ is k-linear since σ is. It is easy to see that

$$[a, bc\rangle = [a, b\rangle c + \sigma(b)[a, c\rangle.$$

In other words, $[a, \cdot\rangle$ is a σ-twisted derivation for each $a \in A$ and algebra endomorphism σ. In fact, $[a, \cdot\rangle$ is called σ*-inner* in analogy with the classical case $\sigma = \mathrm{id}_A$.

From now on we fix $a \in k[\mathsf{G}]$ and assume that σ given by $\sigma(b) := \phi(a, b)b$ is a k-algebra morphism on $k[\mathsf{G}]$ with $\phi : \{a\} \times k[\mathsf{G}] \to \mathrm{Z}(k[\mathsf{G}])$. For $b, c \in k[\mathsf{G}]$ we have

$$0 = \sigma_a(bc) - \sigma_a(b)\sigma_a(c) = (\phi(a, bc) - \phi(a, b)\phi(a, c))bc$$

and so if bc is not a (right) zero divisor $\phi(a, bc) = \phi(a, b)\phi(a, c)$.

We introduce the notation $\phi^{(\ell)}(a, b) := \phi(a, \phi(a, \ldots, \phi(a, b)))$ (ℓ appearances of ϕ). For instance, $\phi^{(3)}(a, b) = \phi(a, \phi(a, \phi(a, b)))$. Also, it is convenient to interpret $\phi^{(0)}(a, b)$ as b.

Lemma 1. *The following identities hold for $b \in k[\mathsf{G}]$:*

(i) $\sigma_a(\phi^{(\ell)}(a, b)) = \phi^{(\ell+1)}(a, b)\phi^{(\ell)}(a, b)$,
(ii) $\sigma_a^{\ell}(b) = \prod_{j=0}^{\ell} \phi^{(\ell-j)}(a, b)^{\binom{\ell}{j}}$.

Proof. Identity (i) follows immediately from definition. The second one is proved by induction where the case $\ell = 1$ is $\sigma_a^1(b) = \sigma_a(b) = \phi(a, b)b$ which is (ii) for $\ell = 1$. Assume now that (ii) holds for ℓ. Then

$$\sigma_a^{\ell+1}(b) = \sigma_a(\sigma_a^{\ell}(b)) = \sigma_a(\prod_{j=0}^{\ell}\phi^{(\ell-j)}(a,b)^{\binom{\ell}{j}}) = \prod_{j=0}^{\ell}\sigma_a(\phi^{(\ell-j)}(a,b))^{\binom{\ell}{j}} =$$
$$= \prod_{j=0}^{\ell}\phi^{(\ell+1-j)}(a,b)^{\binom{\ell}{j}}\phi^{(\ell-j)}(a,b)^{\binom{\ell}{j}} = \prod_{j=0}^{\ell+1}\phi^{(\ell+1-j)}(a,b)^{\binom{\ell+1}{j}},$$

where we have used identity (i) and after re-arranging the product, the Pascal identity $\binom{\ell}{j}+\binom{\ell}{j+1}=\binom{\ell+1}{j+1}$. (Notice that we used that $\phi^{(i)}(a,b)\in \mathrm{Z}(A)$ and that σ_a is multiplicative.) □

Lemma 2. *For $a \in k[\mathsf{G}]$ we have $\phi(a,a)\Delta\circ\sigma = \sigma\circ\Delta$.*

Proof. This follows from the following simple computation:

$$\sigma\circ\Delta(b) = \sigma(ab-\sigma(b)a) = \sigma(a)\sigma(b)-\sigma(\sigma(b))\sigma(a) =$$
$$= \phi(a,a)a\sigma(b)-\sigma(\sigma(b))\phi(a,a)a = \phi(a,a)(a\sigma(b)-\sigma(\sigma(b))a) =$$
$$= \phi(a,a)\Delta\circ\sigma(b).$$

This completes the proof. □

Compare this with [2] wherein we have the reversed order, i.e., $\Delta\circ\sigma = \delta\sigma\circ\Delta$, for $\delta\in A$ (in [2] A was supposed to be commutative as well). In fact, adopting the order from the above Lemma in [2] leads to same result and so we have a connection to the theory developed in [2].

2.1 Main result

Assume that k is an integral domain and let Σ denote the maximal subalgebra of $\mathrm{Z}(k[\mathsf{G}])$ such that $\sigma_a|_{\Sigma} = \mathrm{id}_A$ and such that Σ is an integral domain as well. From now on (unless stated otherwise) we suppose $\phi : \{a\}\times k[\mathsf{G}]\to\Sigma$. This implies that if $s\in\Sigma$ then $\phi(a,s) = \mathbf{1}$ since, on the one hand, $\sigma_a(s) = s$, and on the other, $\sigma_a(s) = \phi(a,s)s$. Also, by construction σ_a satisfies $\sigma_a(sb) = s\sigma_a(b)$ for $s\in\Sigma$. This is all sufficient to have $\Delta(\sigma_a(b)) = \Delta(\phi(a,b)b) = \phi(a,b)\Delta(b)$, for instance. In general $\Delta(sb) = s\Delta(b)$ for $s\in\Sigma$.

Let $a,b\in k[\mathsf{G}]$ and put $\varepsilon_a := \phi(a,a)$ and $\varepsilon_b := \phi(a,b)$. Formally, for $q\in\Sigma^* := \Sigma\setminus\{0\}$, we denote by $\{n\}_q\in\Sigma$ the polynomial $\mathbf{1}+q+q^2+\cdots+q^{n-1}$ for $n\in\mathbb{N}^+ := \mathbb{N}\cup\{0\}$, defining $\{0\}_q := 0$. Note that we do not exclude the possibility of $\{\ell\}_q = \mathbf{1}+q+q^2+\cdots+q^{\ell-1}$ being zero for some $\ell\in\mathbb{N}^+$. Define the "q-binomial coefficient" as the (unique) solution to the "q-Pascal recurrence relation":

$$\binom{n+1}{j+1}_q = q^{n-j}\binom{n}{j}_q + \binom{n}{j+1}_q \tag{3}$$

or 0 either if $j+1<0$ or $j+1>n+1$ and 1 if $j+1=0$ or $j+1=n+1$. It can be proven [3] that $\binom{n}{j}_q$ is a polynomial in q for all n and j. Also, in analogy with the classical case, it can be shown that if neither of the involved products in the denominator is zero, we have $\binom{n}{j}_q := \frac{\{n\}_q!}{\{j\}_q!\{n-j\}_q!}$.

An element $q \in \Sigma^*$ is an n-th root of unity if $q^n = \mathbf{1}$ and a primitive n-th root of unity if $q^n = \mathbf{1}$, and $\{\ell\}_q \neq 0$ for $\ell < n$. Since Σ is a domain, q being an n-th root of unity, i.e., $q^n - \mathbf{1} = 0$, is equivalent to

$$(\mathbf{1} + q + q^2 + \cdots + q^{n-1})(q - \mathbf{1}) = \{n\}_q(q - \mathbf{1}) = 0.$$

So, if $q \neq \mathbf{1}$, $\{n\}_q = 0$.

Proposition 1. *For $a, b \in k[\mathsf{G}]$ we have*

$$\Delta^\ell(b) = \sum_{j=0}^{\ell} (-1)^j \varepsilon_a^{\frac{j(j-1)}{2}} \varepsilon_b^j \binom{\ell}{j}_{\varepsilon_a} a^{\ell-j} b a^j. \tag{4}$$

Proof. The Proposition is verified for $\ell = 1, 2, 3$ without difficulty. Assume that (4) is true for ℓ. Then

$$\Delta^{\ell+1}(b) = \Delta(\Delta^\ell(b)) = \sum_{j=0}^{\ell} (-1)^j \varepsilon_a^{\frac{j(j-1)}{2}} \varepsilon_b^j \binom{\ell}{j}_{\varepsilon_a} \Delta(a^{\ell-j} b a^j). \tag{5}$$

We have

$$\begin{aligned} \Delta(a^{\ell-j} b a^j) = [a, a^{\ell-j} b a^j\rangle = a^{\ell-j+1} b a^j - \sigma_a(a)^{\ell-j} \sigma_a(b) \sigma_a(a)^j a = \\ = a^{\ell-j+1} b a^j - \varepsilon_a^\ell \varepsilon_b a^{\ell-j} b a^{j+1}. \end{aligned}$$

This means that

$$\begin{aligned} \Delta^{\ell+1}(b) = \sum_{j=0}^{\ell} (-1)^j \varepsilon_a^{\frac{j(j-1)}{2}} \varepsilon_b^j \binom{\ell}{j}_{\varepsilon_a} a^{\ell-j+1} b a^j + \\ + \sum_{j=0}^{\ell} (-1)^{j+1} \varepsilon_a^{\frac{j(j-1)}{2}+\ell} \varepsilon_b^{j+1} \binom{\ell}{j}_{\varepsilon_a} a^{\ell-j} b a^{j+1}. \end{aligned}$$

Write the first sum as

$$\sum_{j=1}^{\ell} (-1)^j \varepsilon_a^{\frac{j(j-1)}{2}} \varepsilon_b^j \binom{\ell}{j}_{\varepsilon_a} a^{\ell-j+1} b a^j + a^{\ell+1} b = \mathsf{S}_1 + a^{\ell+1} b$$

and the second as

$$\begin{aligned} \sum_{j=0}^{\ell-1} (-1)^{j+1} \varepsilon_a^{\frac{j(j-1)}{2}+\ell} \varepsilon_b^{j+1} \binom{\ell}{j}_{\varepsilon_a} a^{\ell-j} b a^{j+1} + (-1)^{\ell+1} \varepsilon_a^{\frac{\ell(\ell+1)}{2}} \varepsilon_b^{\ell+1} b a^{\ell+1} = \\ = \mathsf{S}_2 + (-1)^{\ell+1} \varepsilon_a^{\frac{\ell(\ell+1)}{2}} \varepsilon_b^{\ell+1} b a^{\ell+1}. \end{aligned}$$

The S_1-term can be written as

$$\sum_{j=0}^{\ell-1}(-1)^{j+1}\varepsilon_a^{\frac{j(j+1)}{2}}\varepsilon_b^{j+1}\binom{\ell}{j+1}_{\varepsilon_a}a^{\ell-j}ba^{j+1}.$$

Adding S_1 and S_2 we get:

$$\mathsf{S}_1+\mathsf{S}_2=\sum_{j=0}^{\ell-1}(-1)^{j+1}\varepsilon_b^{j+1}\big(\varepsilon_a^{\frac{j(j+1)}{2}}\binom{\ell}{j+1}_{\varepsilon_a}+\varepsilon_a^{\frac{j(j-1)}{2}+\ell}\binom{\ell}{j}_{\varepsilon_a}\big)a^{\ell-j}ba^{j+1}.$$

Note that $\frac{j(j-1)}{2}=\frac{j(j+1)}{2}-j$ so the parentheses becomes

$$\varepsilon_a^{\frac{j(j+1)}{2}}\big(\binom{\ell}{j+1}_{\varepsilon_a}+\varepsilon_a^{\ell-j}\binom{\ell}{j}_{\varepsilon_a}\big).$$

Using (3) this is the same as $\varepsilon_a^{\frac{j(j+1)}{2}}\binom{\ell+1}{j+1}_{\varepsilon_a}$. Then $\mathsf{S}_1+\mathsf{S}_2$ add up to

$$\sum_{j=0}^{\ell-1}(-1)^{j+1}\varepsilon_a^{\frac{j(j+1)}{2}}\varepsilon_b^{j+1}\binom{\ell+1}{j+1}_{\varepsilon_a}a^{\ell-j}ba^{j+1}=$$
$$=\sum_{j=1}^{\ell}(-1)^{j}\varepsilon_a^{\frac{j(j-1)}{2}}\varepsilon_b^{j}\binom{\ell+1}{j}_{\varepsilon_a}a^{\ell+1-j}ba^{j}.$$

Putting everything together yields

$$a^{\ell+1}b+\mathsf{S}_1+\mathsf{S}_2+(-1)^{\ell+1}\varepsilon_a^{\frac{\ell(\ell+1)}{2}}\varepsilon_b^{\ell+1}ba^{\ell+1}=$$
$$=\sum_{j=0}^{\ell+1}(-1)^{j}\varepsilon_a^{\frac{j(j-1)}{2}}\varepsilon_b^{j}\binom{\ell+1}{j}_{\varepsilon_a}a^{\ell+1-j}ba^{j}$$

and the proof is complete.

Suppose ε_a satisfies $\{n\}_{\varepsilon_a}=\mathbf{1}+\varepsilon_a+\varepsilon_a^2+\cdots+\varepsilon_a^{n-1}=0$, that is, $\mathbf{1}\neq\varepsilon_a\in\Sigma\subseteq\mathrm{Z}(k[\mathsf{G}])$ is a primitive n-th root of unity. Then $\binom{n}{j}_{\varepsilon_a}=0$ for $j\neq 0,n$. Hence

$$\Delta^n(b)=a^nb+(-1)^n\varepsilon_a^{\frac{n(n-1)}{2}}\varepsilon_b^nba^n.$$

Assuming that a^n and b commute (if $a^n\in\mathrm{Z}(k[\mathsf{G}])$, for instance), we get

$$\Delta^n(b)=(\mathbf{1}+(-1)^n\varepsilon_a^{\frac{n(n-1)}{2}}\varepsilon_b^n)a^nb.$$

From this follows that

$$\Delta^n(b)=(\mathbf{1}-\varepsilon_b^n)a^nb,$$

if n is odd, and

$$\Delta^n(b)=(\mathbf{1}+(\varepsilon_a^{\frac{n}{2}})^{n-1}\varepsilon_b^n)a^nb,$$

if n is even. However, since ε_a is a primitive n-th root of unity $\varepsilon_a^{\frac{n}{2}} = -\mathbf{1}$ and so both these cases are the same.

Corollary 1. *If, in addition to the above assumptions, $\varepsilon_b^n = \mathbf{1}$ then $\Delta^n(b) = 0$, for all $b \in k[\mathsf{G}]$.*

3 Generalized N-complexes and Examples

A *generalized N-complex*, $N \geq 0$, is a sequence of objects $\{C_i\}_{i\in\mathbb{Z}}$, in an abelian category A together with a sequence of morphisms $\mathrm{d}_i \in \mathrm{Hom}(C_i, C_{i+p})$ for some (fixed) $p \in \mathbb{Z}$ and such that

$$\mathrm{d}^N := \mathrm{d}_{i+(N-1)p} \circ \mathrm{d}_{i+(N-2)p} \circ \cdots \circ \mathrm{d}_{i+p} \circ \mathrm{d}_i = 0 : C_i \to C_{i+Np}.$$

The case $N = 0$ is interpreted as there being no vanishing condition at all on the differential and $N = 1$ means $\mathrm{d} = 0$. We write a generalized N-complex as $(C_n, \mathrm{d}_n)_{n\in\mathbb{Z}}^{N,p}$. If $p = 1$ we get the class of N-complexes and if in addition $N = 2$ we get the ordinary complexes from ordinary homological algebra. Of course we could have defined $\mathrm{d}_i \in \mathrm{Hom}(C_i, C_{i+p_i})$ for some family of p_i's but such a definition would drown in indices so we refrain from explicitly stating it.

In this paper we are considering only the case when $\mathsf{AMod}(k)$, the abelian category of k-(bi-)modules. Also we are mainly concerned with the special case of graded algebras. As a reminder we recall the case of differential graded algebras.

Example 1. Let $D = \bigoplus_{n\in\mathbb{Z}} D_n$ be a graded k-algebra. Then a differential graded structure on D is a k-linear map $\mathrm{d} : D_n \to D_{n+1}$ such that the graded Leibniz rule, $\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{\deg(a)} a\mathrm{d}(b)$, holds for homogeneous $a, b \in D$. This becomes an ordinary 2-complex with $C_n = D_n$.

Note that d is actually a σ-derivation on D with $\sigma(a_n) = (-1)^n a_n$, for $a_n \in D_n$, extended k-linearly on $\mathsf{G} = \cup_{n\in\mathbb{Z}} D_n$, and we have $D = k[\mathsf{G}]$. In fact, σ is linear on each graded component and $\sigma(a_n b_m)$ can be defined (unambiguously) as $(-1)^{n+m} a_n b_m$, for $a_n \in D_n$ and $b_m \in D_m$, hence $\sigma(a_n b_m) = \sigma(a_n)\sigma(b_m)$, so this is well-defined.

Example 2. Generalizing the above example as follows leads to the q-differential graded algebras considered by Abramov [1] among many others. Indeed, let as before $D = \bigoplus_{n\in\mathbb{Z}} D_n$ and take $q \in k$, with the property $q^N = \mathbf{1}$ (usually it is assumed that $k = \mathbb{C}$), and let d be a k-linear map on D such that $\mathrm{d}(ab) = \mathrm{d}(a)b + q^{\deg(a)} a\mathrm{d}(b)$. This is also a σ-derivation on D with $\sigma(a_n) = q^n a_n$ for $a_n \in D_n$ extended k-linearly from $\mathsf{G} = \cup_{n\in\mathbb{Z}} D_n$ to $D = k[\mathsf{G}]$. Clearly the above example is a special case of this one when q is the second root of unity $q = -1$.

3.1 An elaborated example

Here we assume that A is the k-algebra of Laurent polynomials over k, i.e., $A = k[t,t^{-1}]$. This is a $\mathbb{Z}$-graded k-algebra[1] generated over k by $\{\mathbf{1},t,t^{-1}\}$ and so we could either take $\mathsf{G} = \cup_{n\in\mathbb{Z}} kt^n = \cup_{n\in\mathbb{Z}} A_n$, the homogeneous elements, or $\mathsf{G} = \{\mathbf{1},t,t^{-1}\}$ and we would still have $A = k[t,t^{-1}] = k[\mathsf{G}]$. For simplicity we choose $\mathsf{G} = \{\mathbf{1},t,t^{-1}\}$.

The most general σ on G is one on the form $\sigma(t) = q_1 t^{s_1}$ and $\sigma(t^{-1}) = q_2 t^{s_2}$ but this choice have to respect $tt^{-1} = t^{-1}t = \mathbf{1}$ so if σ is multiplicative we have to condition $q_2 = q_1^{-1} =: q$ and $s_2 = -s_1 =: s$. We then have $\sigma(t) = qt^s = \phi(a,t)t$ so $\phi(a,t) = qt^{s-1}$. From this follows $\phi(a,t)\phi(a,t^{-1}) = 1$, i.e., $\phi(a,t)^{-1} = \phi(a,t^{-1})$ by the uniqueness of inverses. Extend σ to A by the obvious $\sigma(u_1t^n + u_2t^m) := u_1\sigma(t^n) + u_2\sigma(t^m)$ for $u_1,u_2 \in k$, $n,m \in \mathbb{Z}$.

Take $a \in A$ and form $\Delta := a(\mathrm{id}_A - \sigma)$. We know that Δ is a σ-derivation since A is commutative. Applying Δ to a homogeneous component A_n we find

$$\Delta(ut^n) = a(\mathrm{id}_A - \sigma)(ut^n) = au(t^n - \phi(a,t)^n t^n) = au(\mathbf{1} - \phi(a,t)^n)t^n.$$

The degree of Δ is therefore in general undefined since $\mathbf{1}$ and $\phi(a,t)^n$ will belong to different graded components; indeed, $\phi(a,t)^n \notin A_0 \approx k$ in general. However, if $\phi(a,t) \in A_0$ then $\phi(a,t)^n \in A_0$ for all $n \in \mathbb{Z}$ since A_0 is a subalgebra. Accordingly, we assume from now on that $\phi(a,t) \in k$. Then $\Delta(ut^n) = au(\mathbf{1} - \phi(a,t)^n)t^n \in A_{n+\deg(a)}$ with $u \in k$.

This means that we have a generalized complex $(A_n,\Delta)_{n\in\mathbb{Z}}^{0,\deg(a)}$, where $\Delta : A_n \to A_{n+\deg(a)}$, for each $a \in A$.

From Proposition 1 we have

$$\Delta^\ell(b) = \sum_{j=0}^{\ell} (-1)^j \phi(a,a)^{\frac{j(j-1)}{2}} \phi(a,b)^j \binom{\ell}{j}_{\phi(a,a)} a^\ell b.$$

Suppose $\phi(a,a)^\ell = \mathbf{1}$ and $\phi(a,a)^m \neq \mathbf{1}$ for $m < \ell$, i.e., $\phi(a,a)$ is a primitive ℓ^{th}-root of unity and suppose $\phi(a,b)^\ell = \mathbf{1}$. Then we are in the situation of Corollary 1:

$$\Delta^\ell(b) = 0, \quad \text{forall} \quad b \in A = k[t,t^{-1}],$$

and so we have constructed an N-complex.

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[1] In fact, if k is a field, then A is actually even a *graded field* in the sense that each homogeneous element is a unit. More to the point in this case: if $a_n \in A_n$ then there is an element $a_{-n} \in A_{-n}$ such that $a_n a_{-n} = a_{-n} a_n = \mathbf{1}$.

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