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Arithmetic geometry of non-commutative spaces with large centres

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Abstract
This paper introduces arithmetic geometry for polynomial identity algebras using non-commutative (formal) deformation theory. Since formal deformation theory is inherently local the arithmetic and geometric results that follow give local information that is not visible when looking at the objects from a commutative angle. For instance, it is a precise meaning to be given to two things being "infinitesimally close", something being obscured from view when restricting only to a commutative algebraic study. A Platonesque way of looking at this is that the commutative world is a "shadow" of a more inclusive non-commutative universe. The present paper aims at laying the foundation for further and deeper study of algebraic number theory and geometry using non-commutative geometry and non-commutative deformation theory. As such we prove no deep theorems. Instead we strive to illustrate, for instance by supplying detailed examples, how non-commutative algebraic geometry can be used to view number-theoretic objects from a different perspective.

Résumé
Cet article présente la géométrie arithmétique pour les algèbres polynomiales identités en utilisant la théorie de la déformation non commutative (formelle). La théorie de la déformation formelle étant intrinsèquement locale, les résultats arithmétiques et géométriques qui en découlent fournissent des informations locales invisibles lorsqu'on observe les objets sous un angle commutatif. Par exemple, il est précis de donner à deux choses un sens «$infiniment\ proche$», ce qui est occulté lorsqu'on se limite à une étude algébrique commutative. Une approche platonicienne consiste à considérer le monde commutatif comme l'«$ombre$» d'un univers non commutatif plus inclusif.
Cet article vise à jeter les bases d'une étude plus approfondie de la théorie algébrique des nombres et de la géométrie, en utilisant la géométrie non commutative et la théorie de la déformation non commutative. À ce titre, nous ne démontrons aucun théorème profond. Au lieu de cela, nous nous efforçons d'illustrer, par exemple en fournissant des exemples détaillés, comment la géométrie algébrique non commutative peut être utilisée pour considérer les objets de la théorie des nombres sous un angle différent.

Keywords Non-commutative arithmetic geometry and deformation theory · Non-commutative rational points · Polynomial identity algebras remark

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1 Introduction

Non-commutative algebra has been present in number theory for a long time (e.g., quaternion algebras and central simple algebras) and as such the approach in this paper is definitely not novel. The novelty presented comes from the use of *non-commutative* algebraic geometry introduced by Laudal [15], based on the *non-commutative* deformation theory he developed in [14].

In addition, deformation theory in arithmetic is also not a novelty. For instance, deformations of Galois representations and deformations of Galois covers of curves in characteristic p are but two examples of prominence. Deformation theory brings out, by its very definition, "local" information and the extension to non-commutative bases further brings out local information that is not visible to a commutative eye. We will see several examples of this later. In fact, "locality" is a fundamental aspect of both arithmetic and geometry.

There are not many papers that are dealing with non-commutative algebraic geometry in the context of arithmetic geometry as far as the author is aware. There is Borek's version of non-commutative Arakelov theory [2, 3] and the recent paper [4] by Chan and Ingalls. Both of these papers use another approach to non-commutative algebraic geometry that is inherently different than the approach taken in the present text. In fact, the starting point for them is Artin and Zhang's notion of "non-commutative projective schemes". This way of looking at non-commutative geometry is global by its very definition (whereas ours is fundamentally local). However, none of the papers [2–4] use non-commutative deformation theory.

Remark 1.1 I would be remiss if I did not remark that A. Connes and his collaborators M. Marcolli and K. Consani also studies arithmetic in the context of non-commutative geometry. However, this version is not "algebraic". See Connes' webpage for more information.

1.1 The idea

Let us illustrate the basic idea with the following (not so) hypothetical situation. Let X be a scheme over a field k and let G be a group acting on X. In general, the quotient X/G does not exist as a scheme unless, for instance, X is quasi-projective and G finite. However, even for finite quotients of quasi-projective schemes, it is sometimes beneficial to introduce stack structures (e.g., at singularities or branch points) on X/G. For instance, if X is a curve, the resulting quotient is often called a "stacky curve". Assume for simplicity that $X = \operatorname{Spec} A$ for some k-algebra A and that G is finite.

Let $k \subseteq k'$. Then there is a one-to-one correspondence between the orbits of $X(k')$ under G and the elements of $(\operatorname{Spec} A^G)(k')$. Let $\boldsymbol{p}$ be a k'-point on X and let $\underline{\boldsymbol{o}}(\boldsymbol{p}) = \operatorname{Spec} A/I$ be the orbit. Then A/I is both a G-module and an A-module; hence an $A[G]$-module. Therefore, we have an identification of simple $A[G]$-modules over k' and points in $(\operatorname{Spec} A^G)(k')$, i.e., the orbits. Observe that $A[G]$ is a commutative ring if and only if G is commutative.

However, instead of looking at $A[G]$ we can look at the crossed product algebra (or skew group ring) $A\langle G\rangle$ (sometimes denoted $A * G$) which, as an abelian group, is the same as $A[G]$ but with multiplication defined by $\sigma x = \sigma(x)\sigma$, for $\sigma \in G$ and $x \in A$. This is obviously a non-commutative ring. It is quite easy to see that there is still a one-to-one correspondence between points of $(\operatorname{Spec} A^G)(k')$ and simple $A\langle G\rangle$-modules over k'. Put, for simplicity, $B := A\langle G\rangle$.

Now, points on $X(k')$ with non-trivial stabiliser (i.e., points where $\sigma x = x$ for all $\sigma \in \mathfrak{h} \subseteq G$; the group $\mathfrak{h}$ is the *stabiliser group* or *isotropy group*) are points of special interest since this is where possible "stacky-ness" comes in. In fact, these are the ramification points of the cover $X \twoheadrightarrow X/G$.

There is a canonical isomorphism between the tangent space at a point $\boldsymbol{p} \in X/G(k')$ and the vector space $\operatorname{Ext}^1_B(M, M)$, where M is the, to $\boldsymbol{p}$, associated B-module. In particular, if $\dim(\operatorname{Ext}^1_B(M, M)) = \dim(X) = \dim(X/G)$, $\boldsymbol{p}$ is a non-singular point. If $\boldsymbol{p}$ is singular on the other hand,

$$\dim(X) = \dim(X/G) \leq \dim(\operatorname{Ext}^1_B(M, M)).$$

However, when A is non-commutative the Ext-dimension can be greater than what can be expected just by looking from a commutative perspective. In other words, we can have

$$\dim(\operatorname{Ext}^1_{A[G]}(M, M)) < \dim(\operatorname{Ext}^1_{A\langle G\rangle}(M, M)).$$

This phenomenon can appear only at points of ramification. The dimension is maximal when the ramification is wild.

Assume now that $\boldsymbol{p}$ and $\boldsymbol{q}$ are two ramification points on X/G with associated B-modules M and N. Then, as modules over $A[G]$ we have $\operatorname{Ext}^1_{A[G]}(M, N) = 0$. However, as modules over $B = A\langle G\rangle$ we always have $\operatorname{Ext}^1_B(M, N) \neq 0$. The interpretation here is that the points $\boldsymbol{p}$ and $\boldsymbol{q}$ lie "infinitesimally close" and that $\operatorname{Ext}^1_B(M, N)$ is the "tangent space between M and N". This is not symmetric. Indeed, we might have

$$\dim(\operatorname{Ext}^1_B(M, N)) \neq \dim(\operatorname{Ext}^1_B(N, M)).$$

In other words, "closeness cares about direction"; $\boldsymbol{p}$ can be "closer" to $\boldsymbol{q}$ than $\boldsymbol{q}$ is to $\boldsymbol{p}$. We will see an example of this in the last section.

Therefore, we can view Ext^1 as a measure of how "ramified" something is, a statement that will be made more precise later in the paper.

The ring $B = A\langle G\rangle$ is an example of a *non-commutative crepant resolution* of A^G (see for instance, [22]). Namely, even if $X/G = \operatorname{Spec} A^G$ have singularities, the ring B has natural regularity properties as a non-commutative ring (see Sect. 7.1). Also, T. Stafford and M. van den Bergh proves in [22] that if A^G has a non-commutative *crepant* resolution, $\operatorname{Spec} A^G$ has rational singularities. Regardless of the "crepant-ness", it is natural to view B as a non-commutative resolution of $\operatorname{Spec} A^G$ as B in any case retains many regularity properties.

1.2 Deformation theory

Let $\mathsf{Mod}\, A$ be the category of left A-modules, $M \in \mathsf{Mod}\, A$ and let $\underline{\mathtt{Def}}_M$ be a deformation functor of M from the category C of local, complete, artinian rings to the category of sets. Then, under some mild conditions (see Sect. 2.2), $\underline{\mathtt{Def}}_M$ has a pro-representing hull $\hat{H}$ that is constructed using the tangent space $\operatorname{Ext}^1_A(M, M)$ and matric Massey products (see Sect. 2.2 or [5], for more details). In other words, $\hat{H}$ is the completion of the local ring of a moduli space of A-modules. Of course, such a moduli space might not exist. However, $\hat{H}$ always does.

Let $\mathsf{N} := \{N_1, N_2, \ldots, N_n\}$ be a family of A-modules. Extending the base category C to a suitable class of non-commutative rings, we can define a deformation functor $\underline{\mathtt{Def}}^{\mathrm{nc}}_{\mathsf{N}}$ of the *family* N encoding the *simultaneous* deformations of the modules N_i. This functor also have a pro-representing hull, but now this is a matrix ring $(\hat{H}_{ij})$ with entries being quotients of non-commutative formal power series rings along the diagonal, and bi-modules off-diagonal. The diagonal comes from the spaces $\operatorname{Ext}^1_A(N_i, N_i)$ and the entries off the diagonal come from $\operatorname{Ext}^1_A(N_i, N_j)$, $i \neq j$. As in the commutative case in the previous paragraph, there is an algorithm to compute $(\hat{H}_{ij})$ (once again, see [5]).

From $(\hat{H}_{ij})$ one constructs the versal family of representations of A

$$\hat{\varrho} : A \to \hat{\mathcal{O}}_{\mathsf{N}},$$

with $\hat{\mathcal{O}}_{\mathsf{N}}$ the matrix ring

$$\hat{\mathcal{O}}_{\mathsf{N}} := \big(\operatorname{Hom}_k(N_i, N_j) \otimes_k \hat{H}_{ij}\big).$$

For more details see Sect. 2.2.

It is, ultimately, the ring $\hat{\mathcal{O}}_{\mathsf{N}}$ that is the local object that we're after and that includes all the local non-commutative information. Unfortunately, this ring is almost always extremely difficult to compute explicitly. We will see a couple of examples where it is actually possible. Easier, but still very hard, is it to compute the tangent spaces $\operatorname{Ext}^1_A(N_i, N_j)$ (which is the first step in computing $\hat{\mathcal{O}}$).

Let us continue the meta-example above. If $X \twoheadrightarrow X/G$ is a singular G-cover with singular point $\boldsymbol{p}$. This is also a branch point. Let M be the G-orbit of $\boldsymbol{p}$, considered as an $A\langle G\rangle$-module. It turns out that $\hat{\mathcal{O}}_M$ captures a lot of the ramification properties of the cover X/G at $\boldsymbol{p}$. For instance, in all examples the author is aware of, the tangent space $\hat{\mathcal{O}}_M$ is significantly different when the ramification is wild. The author is convinced that this is a general phenomenon but he has not studied it to the point where he can come up with a specific result (other than in special cases) or make a general conjecture. The examples that are computed in the paper clearly show this phenomenon.

As a consequence, the ring object $\hat{\mathcal{O}}_M$ will almost certainly include information concerning wild ramification that is not visible through commutative means. In fact, wild ramification of quotient singularities of arithmetic surfaces has attracted some interest in the last decade (see for instance [9, 12, 17]) and it is my hope that the introduction of $\hat{\mathcal{O}}$ will shed new light on wildly ramified singularities. This seems quite natural since $A\langle G\rangle$ can be viewed as a non-commutative resolution[1] of A^G.

1.3 Polynomial identity algebras

Let R be a commutative ring and A an R-algebra. Then A is a *polynomial identity algebra* (*PI-algebra*) if there is a monic $P(\mathbf{x}) \in \mathbb{Z}\langle \mathbf{x}\rangle = \mathbb{Z}\langle x_1, x_2, \dots, x_n\rangle$ such that $P(a_1, a_2, \dots, a_n) = 0$ for all n-tuples $(a_1, a_2, \dots, a_n) \in A^n$. The relation between A and all polynomial identities it satisfies, is quite complicated.

Clearly, commutative algebras are polynomial identity algebras. Furthermore, central simple algebras and Azumaya algebras are PI. Let us look at some more non-trivial examples that have a undeniable arithmetic flavour.

Let R be a commutative domain and A an R-algebra of finite rank over R. We denote by $\operatorname{Fr}(R)$ the field of fractions of R. Then an R-*order* in A is an R-subalgebra $\Lambda \subset A$ such that $A \otimes_R \operatorname{Fr}(R) \simeq \Lambda \otimes_R \operatorname{Fr}(R)$. As an explicit example, we mention orders in quaternion algebras. All R-orders in an R-algebra of finite R-rank, are PI-algebras. The important

[1] These are almost certainly not *crepant* resolutions. There seems to be issues concerning crepant resolutions (even in the commutative case) in mixed characteristic.

observation here is that the module structure of orders in algebras are closely linked to their ideal arithmetic. Since we will be interested in deformation of modules, we see a direct link between our approach to non-commutative geometry and the arithmetic of such orders.

The above examples are incorporated in the larger class of algebras that are finite as modules over their centres. In particular, when G is a finite group, $A\langle G\rangle$ is finite as a module over its centre A^G, and hence a PI-algebra.[2] Therefore, non-commutative resolutions of singularities as (loosely) defined above, are PI-algebras.

Polynomial identity algebras enjoy some remarkable properties and are similar in many respects to commutative algebras. During the 70's and 80's there was a flurry of research done on PI-algebras, and as a consequence a lot is known about these algebras. For instance, many of these algebras have good homological properties (regularity, Cohen–Macaulay-ness, e.t.c.) and localisation works in a way very similar to commutative localisation. It is impossible to give an exhaustive list of papers dealing with these things, but the (quite old) [23] and Chapter 6 of the book [20], might give some insight. A more modern perspective can be found in [16], where PI-algebras are studied using quiver techniques. In fact, the some of the techniques used in [16] can probably be adapted for use in studying the ring object $\hat{\mathcal{O}}$.

Since this paper deals with algebras with "large centres", meaning exactly that the algebras are finite over its centre, all the machinery of PI-algebras are at our disposal. The case where the centre is not "large", or more generally, when the algebras are not PI-algebras, is certainly very interesting, but also significantly more difficult to study from an arithmetic–geometric perspective.

Let A be a PI-algebra with centre $\mathfrak{Z}(A)$ and let $\mathfrak{m}$ be a maximal ideal in $\mathfrak{Z}(A)$. Then the fibre $A\otimes_{\mathfrak{Z}(A)} k(\mathfrak{m})$ over $\mathfrak{m}$ is either a central simple algebra, or not. The subset of all $\mathfrak{m}\in \operatorname{Specm}\mathfrak{Z}(A)$ where $A\otimes_{\mathfrak{Z}(A)} k(\mathfrak{m})$ is a central simple algebra, is called the *Azumaya locus*, $\mathtt{azu}(A)$, and its complement, $\mathtt{ram}(A)$, is the support of a Cartier divisor in $\operatorname{Spec}\mathfrak{Z}(A)$, the *ramification locus*.

The point is the following. Let $\mathfrak{m}\in\mathtt{azu}(A)$. Then $\mathfrak{m}$ is still maximal as an ideal in A. However, if $\mathfrak{m}\in\mathtt{ram}(A)$, then $\mathfrak{m}$ splits into maximal ideals $\mathfrak{m}_i$ inside A. In addition, every ideal in A intersects the centre in a unique ideal and, furthermore, if the ideal is maximal so is the intersected ideal. Hence, over $\mathtt{azu}(A)$ there is a one-to-one correspondence between maximal ideals in $\mathfrak{Z}(A)$ and maximal ideals in A. As a consequence, over $\mathtt{azu}(A)$, A is geometrically the "same" as $\mathfrak{Z}(A)$, the difference being that points $\mathfrak{m}\in\mathtt{azu}(A)$, increases in dimension when viewed inside A.

This indicates that the interesting part, at least from a non-commutative view-point, of $\mathfrak{Z}(A)$ is the ramification locus, and this is indeed the case. Let $\mathfrak{m}\in\mathtt{ram}(A)$. Then, for some $n\geq 1$, $\mathfrak{m}=\mathfrak{m}_1\cdots\mathfrak{m}_n$ as ideals in A. Let $N_i:=A/\mathfrak{m}_i$, considered as left A-modules. Then $\operatorname{Ext}^1_A(N_i,N_j)\neq 0$. In fact, we have the equivalence

$$\operatorname{Ext}^1_A(N_i,N_j)\neq 0 \iff \mathfrak{m}_i\cap\mathfrak{Z}(A)=\mathfrak{m}_j\cap\mathfrak{Z}(A).$$

This is the content of Müller's theorem (see Sect. 4). Expressed differently, in geometric terms, the points in A over a point $\mathfrak{m}\in\mathtt{ram}(A)$, are infinitesimally close. Here is then where the ring object $\hat{\mathcal{O}}$ enters non-trivially.

One of the important aspects of the above is the relation to the Brauer group of the centre. Indeed, any $\mathfrak{m}\in\mathtt{azu}(A)\subset\operatorname{Spec}\mathfrak{Z}(A)$, determines a unique element in $\operatorname{Br}\mathfrak{Z}(A)$, namely $A\otimes_{\mathfrak{Z}(A)}\mathfrak{Z}(A)_{\mathfrak{m}}$; reducing this algebra modulo $\mathfrak{m}$, gives $A\otimes_{\mathfrak{Z}(A)} k(\mathfrak{m})$, an element in $\operatorname{Br} k(\mathfrak{m})$. We will make a few more comments on Brauer groups in Sect. 4.

[2] Note that, depending on R and A, $A\langle G\rangle$ can be an order in some algebra (that need not be unique).

Allowing ourself to make a definite proposition, the way to think about the above is to view A as a "non-commutative thickening" of $\mathfrak{Z}(A)$ or, as suggested in the abstract, that $\mathfrak{Z}(A)$ is a commutative "shadow" of A, referring to Plato's cave allegory.

1.4 Vistas

There are three, very much interconnected, reasons why the author started thinking about the contents of the following paper.

(1) To study degenerations in moduli spaces of abelian varieties using non-commutative algebraic geometry.
(2) Using non-commutative algebraic geometry to study arithmetic quotient singularities and G-covers of arithmetic schemes in non-tame cases (for instance when the cover is wild or ferocious).
(3) As an initial, somewhat approachable, class of examples before tackling spaces that are not necessarily finite over their centres.

It would be too lengthy to say anything about the first point. The other two points will hopefully be amply motivated throughout the paper. Maybe the reader can infer the idea for (1) by studying this paper.

1.5 Organisation

The paper is organised as follows. Section 2.1 introduces a first tentative notion of non-commutative space that we will use. This definition will be expanded upon in Sect. 2.3. Section 2.2 gives a short survey on non-commutative deformation theory. This section can probably be skimmed and referred to as needed. It includes the definition of the ring objects $\hat{\mathcal{O}}$.

The next section, Sect. 2.3, defines non-commutative algebraic spaces, which will be the main underlying object in all that follows. Following this is the important Sect. 3 that introduces rational points on non-commutative algebraic spaces. As said above, PI-algebras play a central role in this paper and Sect. 4 discusses the non-commutative geometry of these algebras. In this section, we study rational points and their associated Brauer classes, the tangent structure of rational points and the central topology. Furthermore, we introduce in Sect. 4, invertible sheaves and Picard groups, as well as morphisms between non-commutative spaces. The final subsection of Sect. 4 discusses non-commutative subspaces. If one were to pick a "main theorem" of this paper, it would be Theorem 4.3 in Sect. 4.4. This theorem gives a complete description of the (non-commutative) infinitesimal structure of rational points on a space that is finite over its centre.

The next part of the paper is devoted to non-commutative Diophantine Geometry. This section starts of with a recollection of height functions and then we define three types of non-commutative versions: one "central", one that is representation-theoretic, and one that is "infinitesimal", taking into consideration the tangent structures of the rational points.

Section 6 starts by extending the algebras and spaces to "infinite fibres", mimicking the technique used in Arakelov geometry. We will not develop a fully fledged Arakelov theory involving metrised Hermitian vector bundles and Chow groups and so on (this should definitely be done!). However, the extension to infinite fibres has the great benefit of packaging the geometric and arithmetic properties into one coherent object. In this way, the dual nature (geometric/arithmetic) of PI-algebras becomes immediately visible. This section introduces preliminary versions of invertible sheaves, Cartier and Weil divisors as well as a rudimentary intersection theory of divisors. It should be emphasised that these notions are preliminary, and they should be refined in the future in order to be truly useful.

Finally, in Sect. 7 we look at three examples. The first is the quotient $\mathbb{A}^2/\mu_3$ over an order $\mathfrak{o}$ in a number field, such that $\mathbb{Z}[\zeta_3] \subseteq \mathfrak{o}$. This quotient is defined by $x \mapsto \zeta_3 x$ and $y \mapsto \zeta_3 y$, where ζ_3 is a third root of unity. As is well-known

$$\mathbb{A}^2/\mu_3 \simeq \operatorname{Spec}\left(\frac{\mathfrak{o}[u, v, w]}{(w^3 - uv)}\right),$$

at least over fibres where 3 is invertible. We compute the tangent spaces over all ramification points and compute $\hat{\mathcal{O}}$, at least up to obstructions of order two and we give the non-commutative thickening of $\mathbb{A}^2/\mu_3$. We also discuss some arithmetic properties of this thickening such as divisors and heights. As mentioned before, computations here are difficult, so, as to not overstate this paper's importance and let it expand beyond reasonable bounds, more difficult computations will have to wait for another time.

The second example is a family of degree-two thickenings of the $\mathbb{Z}$. We look at the degree-two cover $\mathbb{Z}[\sqrt{d}]/\mathbb{Z}$, where $d \equiv 2, 3 \pmod 4$ (and where the integer d is assumed square-free). The quotient of $\operatorname{Spec}\mathbb{Z}[\sqrt{d}]$ by its Galois group $G = \mathbb{Z}/2$ is $\operatorname{Spec}\mathbb{Z}$ and so the orbits are in one-to-one correspondence with primes in $\mathbb{Z}$. Therefore, it is reasonable to look at the simple $\mathbb{Z}[\sqrt{d}]\langle G\rangle$-modules, which are also in a one-to-one correspondence with $\operatorname{Spec}\mathbb{Z}$. From this we can construct a family of non-commutative spaces, parametrised by d, which can be viewed as non-commutative thickenings of $\mathbb{Z}$. In fact, the tangent structure is trivial over all unramified points, but over the ramified ones the ring object $\hat{\mathcal{O}}$ is non-trivial. In addition, in the case where the ramification is wild (i.e., at all places over 2), the ring is also obstructed in the sense that the underlying deformations are obstructed.

In the last example we consider an order over an arithmetic surface. This is algebra is not constructed as a quotient of a scheme by a group. Instead, this is constructed by considering a quantum plane over the surface and then factoring out by an ideal. The resulting tangent structure turns out to be quite interesting. Indeed, we are in the situation alluded to above, where a point "$\boldsymbol{p}$ is closer to $\boldsymbol{q}$ than $\boldsymbol{q}$ is to $\boldsymbol{p}$". The way this phenomenon manifests itself is that, in this case, $\operatorname{Ext}^1_A(\boldsymbol{p}, \boldsymbol{q}) = k^2$, but $\operatorname{Ext}^1_A(\boldsymbol{q}, \boldsymbol{p}) = k$. An arithmetic study of this situation is very interesting and is probably worth investigating further.

Two final remarks

- I have made the conscious decision to not be consistent in notation and language. The biggest crime is in using the word *centre* of an algebra, both as an algebra itself, but also as a commutative scheme. Therefore, the following typical phrase "let $\mathfrak{m}$ be a point on $\mathfrak{Z}(A)$" will appear frequently. But not only that: we will use "maximal ideal" and "point" interchangeably. Later we will also view structure morphisms of representations as "points". I will make the brazen assumption that this will not cause the reader too much headache.
- We will switch between global constructions and affine construction rather freely. Where the global situation (i.e., where the base is a scheme) is not a hindrance we will use this. However, at certain points, using algebras over general base schemes, is notationally unwieldy and often obscures the underlying idea by introducing unnecessary complexity in language. In those cases, we will unabashedly work over affine patches. I'm reasonably certain that everything can be globalised straightforwardly, or at least without too much effort.

Notation

The following notation will be adhered to throughout.

- The notation $\mathfrak{o}$ will denote a general base ring.
- For a commutative algebra R we denote (Com_R) the category of commutative R-algebras, and (Alg_R) the category of, not necessarily commutative, R-algebras.
- The category of A-modules will be denoted (Mod_A).
- $\mathsf{Mod}\, A$ will denote the *groupoid* of left A-modules up to isomorphism.
- The ring of $n \times n$-matrices over R is denoted $M_n(R)$.
- The word "ideal" will always mean *2-sided* ideal.
- The notation Max A denotes the *set* of (2-sided) maximal ideals, while Specm A denotes the maximal spectrum of A, i.e., Max A together with the Zariski–Jacobson topology (see Sect. 2).
- All modules are *left* modules unless otherwise explicitly specified.
- $\mathfrak{Z}(A)$ denotes the centre of A.
- For $\mathfrak{p}$ a prime in A, $k(\mathfrak{p})$ denotes the residue class field of $\mathfrak{p}$.
- The algebraic closure of a field k is denoted k^{al}.
- Abelian sheaves are denoted with scripted letters.
- All schemes and algebras are noetherian. Schemes are also assumed to be separated.

2 Non-commutative algebraic spaces

2.1 Non-commutative spaces

Let X be a scheme and let $\mathscr{A}$ be a coherent $\mathscr{O}_X$-algebra, with $\mathscr{O}_X \subseteq \mathfrak{Z}(\mathscr{A})$.

Definition 2.1 The $\mathscr{A}$-module $\mathscr{M}$, with structure morphism

$$\varrho : \mathscr{A} \to \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{M}),$$

is *simple* if $\mathscr{M}$ has no non-trivial $\mathscr{A}$-submodules. Since localisation is flat, this implies that $\mathscr{M}$ is simple on the stalks, i.e., for all $\boldsymbol{p} \in X$,

$$\mathscr{M}_{\boldsymbol{p}} := \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{O}_{X,\boldsymbol{p}}$$

is simple as an $\mathscr{A}_{\boldsymbol{p}} := \mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{O}_{X,\boldsymbol{p}}$-module. From this it follows that the fibres are also simple.

Denote by $\mathsf{Mod}\,\mathscr{A}$ the *set* (or *groupoid*) of all $\mathscr{A}$-modules up to isomorphism, and $\mathsf{Mod}_n\mathscr{A}$ the set of rank-n such. In addition, put $\mathsf{Simp}_n\mathscr{A}$ as the set (or groupoid) of isoclasses of simple $\mathscr{A}$-modules, locally free of rank n (over $\mathscr{O}_X$) and

$$\mathsf{Simp}\,\mathscr{A} := \bigcup_n \mathsf{Simp}_n\mathscr{A},$$

the union over all n.

Remark 2.1 By a theorem of M. Artin (later extended by C. Procesi) the set $\mathsf{Simp}_n A$ of simple A-modules can be endowed with the structure of a *commutative* affine scheme, at least over a field of characteristic zero.

There is a natural topology on $\mathsf{Mod}\,\mathscr{A}$, namely the Zariski–Jacobson topology T_{ZJ}. This is the topology generated by the distinguished opens

$$\begin{aligned} D_f &:= \left\{\mathscr{M} \in \mathsf{Mod}\,\mathscr{A} \;\middle|\; \mathrm{Ann}_f(\mathscr{M}) \neq 0,\ \ f \in \mathscr{A}\right\} \\ &= \left\{\mathscr{M} \in \mathsf{Mod}\,\mathscr{A} \;\middle|\; f \notin \mathrm{Ann}_{\mathscr{A}}(\mathscr{M})\right\}, \end{aligned} \tag{1}$$

w here $\mathrm{Ann}_f(\mathscr{M})$ is the f-annihilator of $\mathscr{M}$, i.e., the submodule of elements $m \in \mathscr{M}$ such that $fm = 0$, and $\mathrm{Ann}_{\mathscr{A}}(\mathscr{M})$ is the $\mathscr{A}$-annihilator, i.e., the ideal in $\mathscr{A}$ of all $a \in \mathscr{A}$ such that $a\mathscr{M} = 0$. Observe that we can, and sometimes will, identify $\mathscr{M}$ with its annihilator ideal $\mathrm{Ann}_{\mathscr{A}}(\mathscr{M})$ in $\mathscr{A}$. In addition, observe that

$$\ker\Big(\mathscr{A} \xrightarrow{\varrho} \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{M})\Big) \subseteq \mathrm{Ann}_{\mathscr{A}}(\mathscr{M}),$$

where ϱ is the structure morphism of the $\mathscr{A}$-module $\mathscr{M}$.

We have that $D_f \cap D_g = D_{fg}$ and $D_f \cap D_g = D_g \cap D_f$, so $D_{fg} = D_{gf}$. When $\mathscr{A}$ is commutative we get back the Zariski topology on $\mathbf{Spec}\,\mathscr{A}$ (the global spectrum).

Remark 2.2 If X is an S-scheme, we will assume that all $\mathscr{A}$-modules are of finite rank as $\mathscr{O}_S$-modules. In other words, if $f : X \to S$ is the structure morphism, we assume that $f_*\mathscr{M}$ is a locally free $f_*\mathscr{A}$-module of finite rank on S. So, for instance, if $X = \mathrm{Spec}\,B$, $S = \mathrm{Spec}\,K$ (for K a field) and A a B-algebra, then an A-module M over $X = \mathrm{Spec}\,B$, needs to be a finite-dimensional K-vector space and where A acts on M as a K-algebra.

Now, let

$$\mathsf{M} := \left\{\mathscr{M}_1, \mathscr{M}_2, \ldots, \mathscr{M}_\ell\right\}$$

be a family of coherent $\mathscr{A}$-modules (that are finite over S as mentioned in Remark 2.2) such that

$$\mathrm{rk}_{\mathscr{O}_X}\big(\mathscr{E}xt^1_{\mathscr{A}}(\mathscr{M}_i, \mathscr{M}_j)\big) < \infty,$$

for all $1 \leq i, j \leq \ell$.

The *tangent space* at M is the set

$$\mathbf{T}_{\mathsf{M}} := \left\{\mathscr{E}xt^1_{\mathscr{A}}(\mathscr{M}_i, \mathscr{M}_j) \mid 1 \leq i, j \leq \ell\right\},$$

of $\mathscr{O}_S$-modules. Associated to M is the *tangent space graph* defined by taking as the set of vertices the set M. There are n directed edges from $\mathscr{M}_i$ to $\mathscr{M}_j$ if

$$\mathrm{rk}_{\mathscr{O}_S}\big(\mathscr{E}xt^1_{\mathscr{A}}(\mathscr{M}_i, \mathscr{M}_j)\big) = n.$$

Observe that there is not necessarily any symmetry in shifting places of $\mathscr{M}_i$ and $\mathscr{M}_j$. Indeed, for $i \neq j$,

$$\mathrm{rk}_{\mathscr{O}_S}\big(\mathscr{E}xt^1_{\mathscr{A}}(\mathscr{M}_i, \mathscr{M}_j)\big) \neq \mathrm{rk}_{\mathscr{O}_S}\big(\mathscr{E}xt^1_{\mathscr{A}}(\mathscr{M}_j, \mathscr{M}_i)\big),$$

in general.

Finally, let S/R be an arbitrary ring extension of commutative rings and let M and N be free of finite rank over R. Then the isomorphism (e.g., [13, Lemma 7.4])

$$\mathrm{Hom}_{A\otimes_R S}(M \otimes_R S, N \otimes_R S) \simeq \mathrm{Hom}_A(M, N) \otimes_R S,$$

implies that

$$\mathrm{Ext}^{\bullet}_{A\otimes_R S}(M \otimes_R S, N \otimes_R S) \simeq \mathrm{Ext}^{\bullet}_A(M, N) \otimes_R S.$$

From this follows that

$$\mathrm{rk}_S\big(\mathrm{Ext}^{\bullet}_{A\otimes_R S}(M\otimes_R S, N\otimes_R S)\big)=\mathrm{rk}_R\big(\mathrm{Ext}^{\bullet}_A(M,N)\big),$$

so the dimension of $\mathrm{Ext}^1_A(M,N)$ is constant for base extensions. Therefore, the dimensions in the augmented tangent space $\mathbf{T}^{\bullet}_{(M,N)}$ are also constant under base change. Clearly, this globalises.

2.2 Non-commutative deformation theory of modules

Let A be a (not necessarily commutative) k-algebra, where k is a commutative ring, and let (Mod_A) be a k-linear abelian category of *left* A-modules. We recall that k-linear means that every Hom-set is a k-module.

Definition 2.2 Let Λ be a k-algebra. The category $(\mathrm{Mod}_A)_\Lambda$ of *right* Λ*-objects* is the category of pairs (Y,ϱ) where $X\in(\mathrm{Mod}_A)$ and $\varrho:\Lambda\to\mathrm{End}(Y)$ and where $\varrho(\Lambda)$ acts on the *right* on Y. The morphisms are the obvious commutative diagrams.

The Λ-object (Y,ϱ) is Λ*-flat* if the functor

$$Y\otimes_\Lambda - : (\mathrm{Mod}_\Lambda)\to(\mathrm{Mod}_A)_\Lambda$$

is exact.

Let $(\mathrm{art}_k)_r$, $r>0$, be the category whose objects are morphisms

$$k^r\to\Lambda\overset{\alpha}{\to}k^r,$$

such that the composition is the identity on k^r and such that $J:=\ker(\alpha)$ is nilpotent. Morphisms are the obvious commutative diagrams.

If $\{e_1,\dots,e_r\}$ are the idempotents of k^r, then we put $\Lambda_{ij}:=e_i\Lambda e_j$. The diagonal consists of subalgebras of Λ and the entries off the diagonal are Λ-bimodules. Notice that $\alpha(\Lambda_{ij})=\delta_{ij}k$ (Kronecker's δ-function).

We define $(\widehat{\mathrm{art}}_k)_r$ to be the category of r-pointed pro-objects associated with $(\mathrm{art}_k)_r$. In other words, an object S in $(\widehat{\mathrm{art}}_k)_r$ is a projective limit

$$S=\varprojlim_n S/J^n,\quad S/J^n\in(\mathrm{art}_k)_r.$$

Here J is the kernel of the morphism $S\to k^r$ (S is by definition r-pointed).

Definition 2.3 Let $\mathsf{N}:=\{N_1,N_2,\dots,N_r\}$ be a family of (left) A-modules. Put $A_\Lambda:=A\otimes_k\Lambda$.

(i) Then a *lifting of* N *to* (Λ,ϱ) is an A_Λ-module N_Λ that is Λ-flat, i.e., that the functor

$$\mathsf{N}_\Lambda\otimes_\Lambda - : (\mathrm{Mod}_\Lambda)\to(\mathrm{Mod}_A)_\Lambda$$

is exact. The flatness implies that, if N_i is free of rank n over k, then $N_i\otimes_k\Lambda$ is also free of rank n as a (right) Λ-module. This means that $\mathsf{N}_\Lambda=\mathsf{N}\otimes_k\Lambda$ as (right) Λ-module and, in addition,

$$\mathsf{N}_\Lambda=\mathsf{N}\otimes_k\Lambda=\big(N_i\otimes_k\Lambda_{ij}\big)=\bigoplus_{i,j=1}^{r}M_i\otimes_k\Lambda_{ij}.$$

(ii) We also require that the special fibre is N, i.e., that there is an isomorphism

$$f_\Lambda : \mathsf{N}_\Lambda = \mathsf{N} \otimes_k \Lambda \xrightarrow{\mathrm{id}\otimes\alpha} \mathsf{N},$$

induced from the morphism $\alpha : \Lambda \to k^r$.

(iii) Two liftings N_Λ and N'_Λ are isomorphic if there is an isomorphism

$$h : \mathsf{N}_\Lambda \to \mathsf{N}'_\Lambda$$

of A_Λ-modules such that

$$f'_\Lambda = (h \otimes \mathrm{id}) \circ f_\Lambda.$$

(iv) There is a *non-commutative deformation functor*

$$\underline{\mathtt{Def}}_{\mathsf{N}} : \ (\mathrm{art}_k)_r \to (\mathrm{Set}), \quad \Lambda \mapsto \underline{\mathtt{Def}}_{\mathsf{N}}(\Lambda),$$

where

$$\underline{\mathtt{Def}}_{\mathsf{N}}(\Lambda) := \Big\{\text{all liftings of } \mathsf{N} \text{ to } \Lambda, \text{ up to isomorphism of liftings}\Big\},$$

and where $\underline{\mathtt{Def}}_{\mathsf{N}}(k^r) = \{\bullet\}$.

Observe that $\underline{\mathtt{Def}}_{\mathsf{N}}(\Lambda)$ is a groupoid.

Let $\underline{\mathtt{Def}}$ be any deformation functor. Any $\Lambda \in (\mathrm{art}_k)_r$ comes with a fixed injection $i : k^r \to \Lambda$ and so $\underline{\mathtt{Def}}$ determines a unique element $\bullet_\Lambda := \underline{\mathtt{Def}}(\bullet) \in \underline{\mathtt{Def}}(\Lambda)$.

In addition, any $\lambda \in \underline{\mathtt{Def}}(\Lambda)$ reduces to $\bullet$ under the surjection $\alpha : \Lambda \to k^r$. Any $\lambda \in \underline{\mathtt{Def}}(\Lambda)$ is a *lift* of $\bullet$ to Λ. The *trivial lift* of $\bullet$ is the element $\bullet_\Lambda \in \underline{\mathtt{Def}}(\Lambda)$.

We can extend the deformation functor from $(\mathrm{art}_k)_r$ to $(\widehat{\mathrm{art}}_k)_r$ by putting

$$\underline{\mathtt{Def}}(\hat{\Lambda}) := \varprojlim_n \underline{\mathtt{Def}}(\hat{\Lambda}/J^n), \quad \hat{\Lambda} \in (\widehat{\mathrm{art}}_k)_r.$$

A *pro-couple* for $\underline{\mathtt{Def}}$ is a pair $(\hat{H}, \xi)$ with $\hat{H} \in (\widehat{\mathrm{art}}_k)_r$ and $\xi \in \underline{\mathtt{Def}}(\hat{H})$. A morphism of pro-couples $(\hat{H}_1, \xi_1)$ and $(\hat{H}_2, \xi_2)$ is (obviously) a morphism $f : \hat{H}_1 \to \hat{H}_2$ such that $\underline{\mathtt{Def}}(f)(\xi_1) = \xi_2$.

Yoneda's lemma in the present context states that

$$\mathrm{Hom}(\underline{\mathrm{h}}_{\hat{H}}, \underline{\mathtt{Def}}) \xrightarrow{\simeq} \underline{\mathtt{Def}}(\hat{H}),$$

where $\underline{\mathrm{h}}_{\hat{H}} := \mathrm{Hom}(\hat{H}, -)$. Therefore, any $\xi \in \underline{\mathtt{Def}}(\hat{H})$ gives a unique morphism of functors $f_\xi : \underline{\mathrm{h}}_{\hat{H}} \to \underline{\mathtt{Def}}$.

If f_ξ is an isomorphism of functors, then $\underline{\mathtt{Def}}$ is *pro-representable* by the *universal pro-couple* $(\hat{H}, \xi)$. This is unique up to unique isomorphism of pro-couples.

Let $f_\xi : \underline{\mathrm{h}}_{\hat{H}} \to \underline{\mathtt{Def}}$ be a morphism of functors satisfying, for any surjective morphism $\Lambda \to \Lambda'$ in $(\mathrm{art}_k)_r$, the property that

$$\underline{\mathrm{h}}_{\hat{H}}(\Lambda) \to \underline{\mathrm{h}}_{\hat{H}}(\Lambda') \underset{\underline{\mathtt{Def}}(\Lambda')}{\times} \underline{\mathtt{Def}}(\Lambda)$$

is a surjective morphism of functors. We then say that $(\hat{H}, \xi)$ is *versal* and that $\hat{H}$ is a *pro-representing hull* with *versal family*, ξ.

It is worth pointing out that the above works, word for word, in any k-linear abelian tensor category.

Theorem 2.1 *Suppose k is a field and let $\mathsf{N} = \{N_1, N_2, \ldots, N_r\}$ be a finite family of A-modules, with $\dim_k(N_i) < \infty$ for all $1 \leq i \leq r$. Assume also that*

$$\dim_k\big(\mathrm{Ext}^i_A(N_i, N_j)\big) < \infty, \quad i = 1, 2.$$

Then there is a pro-representable hull $(\hat{H}_{ij})$ for the deformation functor $\underline{\mathtt{Def}}_{\mathsf{N}}$, with versal family

$$\hat{\mathcal{O}}_{\mathsf{N}} := \mathsf{N} \otimes_k \hat{H} = (N_i \otimes_k \hat{H}_{ij}).$$

The algebra morphism (the reduction to the special fibre)

$$\hat{\mathcal{O}}_{\mathsf{N}} = \mathsf{N} \otimes_k \hat{H} \xrightarrow{\mathrm{id} \otimes \alpha} \mathsf{N} \otimes_k k^r = \mathsf{N}$$

(we identify N and $\mathsf{N} \otimes_k k^r = \oplus N_i$) is implicit in the construction.

Furthermore, there is an algorithm that computes the hull using (matric) Massey products.

We can re-phrase the construction using the structure morphisms of the modules. We begin by noting the isomorphisms

$$\begin{aligned}\mathrm{End}_k(\mathsf{N}) \otimes_k \Lambda \simeq \mathrm{End}_\Lambda(\mathsf{N}_\Lambda) &\simeq \Big(\mathrm{Hom}_k\big(N_i, N_j \otimes_k \Lambda_{ij}\big)\Big) \\ &\simeq \big(\mathrm{Hom}_k(N_i, N_j) \otimes_k \Lambda_{ij}\big).\end{aligned}$$

Now, let

$$\varrho := \oplus \varrho_i : A \longrightarrow \bigoplus_{i=1}^{r} \mathrm{End}_k(N_i) = \mathrm{End}_k(\mathsf{N}) \otimes_k k^r = \mathrm{End}_{k^r}\big(\mathsf{N} \otimes_k k^r\big)$$

be the structure morphism of the family N. Then a *lifting of ϱ to Λ* is an algebra morphism

$$\varrho_\Lambda : A \longrightarrow \mathrm{End}_k(\mathsf{N}) \otimes_k \Lambda = \big(\mathrm{Hom}_k(N_i, N_j) \otimes_k \Lambda_{ij}\big)$$

such that the diagram

$$\begin{tikzcd} & \mathrm{End}_k(\mathsf{N}) \otimes_k \Lambda \arrow[d, "\mathrm{id} \otimes \alpha"] \\ A \arrow[ur, "\varrho_\Lambda"] \arrow[r, "\varrho"'] & \bigoplus_1^r \mathrm{End}_k(N_i) \end{tikzcd}$$

commutes. The vertical morphism is

$$\mathrm{End}_k(\mathsf{N}) \otimes_k \Lambda \xrightarrow{\mathrm{id} \otimes \alpha} \mathrm{End}_k(\mathsf{N}) \otimes_k k^r = \bigoplus_{i=1}^{r} \mathrm{End}_k(N_i).$$

A *deformation* of ϱ is then simply an equivalence class of lifts under equivalence of representations.

From the viewpoint of structure morphisms, the versal family for the corresponding deformation functor $\underline{\mathtt{Def}}_\varrho$ is the morphism

$$\hat{\varrho}_{\mathsf{N}} : A \to \mathrm{End}_k(\mathsf{N}) \otimes_k \hat{H} = \big(\mathrm{Hom}_k(N_i, N_j) \otimes_k \hat{H}_{ij}\big).$$

This way of viewing deformation theory of modules (i.e., via structure morphisms) is clearly equivalent to the first (i.e., via the category (Mod_A)).

Put

$$\hat{\mathcal{O}}_{\mathsf{N}} := \mathrm{End}_k(\mathsf{N}) \otimes_k \hat{H} = \big(\mathrm{Hom}_k(N_i, N_j) \otimes_k \hat{H}_{ij}\big).$$

It is natural to view this as the completed local ring at the family N. Consequently, any algebraisation H of $\hat{H}$ gives a ring morphism

$$\varrho_{\mathsf{N}} : A \longrightarrow \operatorname{End}_k(\mathsf{N}) \otimes_k H = \big(\operatorname{Hom}_k(N_i, N_j) \otimes_k H\big)$$

and it is natural to view

$$\mathfrak{O}_{\mathsf{N}} := \operatorname{End}_k(\mathsf{N}) \otimes_k H = \big(\operatorname{Hom}_k(N_i, N_j) \otimes_k H\big)$$

as the local ring at N. Observe that we cannot claim that algebraisations are unique, so $\mathfrak{O}_{\mathsf{N}}$ is not necessarily uniquely determined by $\hat{\mathfrak{O}}_{\mathsf{N}}$.

For a sub-family $\mathsf{N}_0 \subseteq \mathsf{N}$, we get an, up to isomorphism, canonical restriction morphism

$$\mathfrak{O}_{\mathsf{N}} \xrightarrow{\text{res}} \mathfrak{O}_{\mathsf{N}_0}.$$

This can be used to extend the above definition to infinite families of modules.

Let N be an infinite family of A-modules and let S be the category of simplicial sets, i.e., the category of contravariant functors $F : \Delta \to (\mathrm{Set})$, where Δ denotes the category of simplicies. Put

$$\hat{\mathfrak{O}}_{\mathsf{N}} := \varprojlim_{\mathsf{N}_0 \subseteq \mathsf{N}} \hat{\mathfrak{O}}_{\mathsf{N}_0}, \quad \text{with versal family} \quad \hat{\varrho}_{\mathsf{N}} := \varprojlim_{\mathsf{N}_0 \subseteq \mathsf{N}} \varrho_{\mathsf{N}_0} : A \to \hat{\mathfrak{O}}_{\mathsf{N}}.$$

We now define the following formal ring object

$$\hat{\mathfrak{O}} := \hat{\mathfrak{O}}_{\mathsf{Mod}A} := \varprojlim_{\mathsf{S}} \hat{\mathfrak{O}}_{\mathsf{S}} \qquad \text{with} \qquad \hat{\varrho} := \varprojlim_{\mathsf{S}} \hat{\varrho}_{\mathsf{S}} : A \to \hat{\mathfrak{O}},$$

and its (possibly non-unique) algebraic ring object

$$\mathfrak{O} := \mathfrak{O}_{\mathsf{Mod}A} := \varprojlim_{\mathsf{S}} \mathfrak{O}_{\mathsf{S}} \qquad \text{with} \qquad \varrho := \varprojlim_{\mathsf{S}} \varrho_{\mathsf{S}} : A \to \mathfrak{O}.$$

It should be clear what the notation means.

The exact same construction extends to sheaves over a scheme, with the exception that one needs to use a global version of Hochschild cohomology instead of the ordinary affine one (which gives the Ext-groups). See [5, Sec. 3.4] for details on this.

2.3 Non-commutative algebraic spaces

We now return to the global situation.

Definition 2.4 Let $\mathscr{A}$ be a coherent $\mathcal{O}_X$-algebra as above. We introduce two pairs

$$\mathfrak{X}_{\mathscr{A}} := (\mathsf{Mod}\,\mathscr{A}, \mathfrak{O}), \quad \text{and} \quad \hat{\mathfrak{X}}_{\mathscr{A}} := (\mathsf{Mod}\,\mathscr{A}, \hat{\mathfrak{O}}),$$

where the former is the *non-commutative algebraic space* (or *non-commutative scheme*) associated with $\mathscr{A}$, and the latter the *formal* non-commutative space associated with $\mathscr{A}$. We also put

$$\mathfrak{X}_n := \mathfrak{X}_{\mathscr{A},n} := (\mathsf{Mod}_n\mathscr{A}, \mathfrak{O}_n),$$

and similarly with the formal space. Here $\mathfrak{O}_n$ is obviously the restriction of $\mathfrak{O}$ to $\mathsf{Mod}_n\,\mathscr{A}$. The space $\mathfrak{X}_{\mathscr{A}}$ is given the Zariski–Jacobson topology as defined in Sect. 2.1.

Remark 2.3 Remember that $\mathsf{Mod}\,\mathscr{A}$ is the groupoid of all $\mathscr{A}$-modules *up to isomorphism*.

The algebra $\mathcal{A}$ is to be viewed as the algebra of global sections of $\mathfrak{O}$, i.e., informally, as

$$\mathcal{A} = \Gamma(\mathfrak{X}, \mathfrak{O}) = H^0(\mathfrak{X}, \mathfrak{O}).$$

The object $\mathfrak{O}$ gives the *local* information of $\mathfrak{X}_{\mathcal{A}}$. We call $\mathfrak{O}$ (and $\hat{\mathfrak{O}}$) the *structure object* (*formal* structure object) of $\mathfrak{X}_{\mathcal{A}}$. If $\mathcal{A}$ is implicit in the discussion we often write $\mathfrak{X}$ for $\mathfrak{X}_{\mathcal{A}}$.

The local nature of $\mathfrak{O}$ is the reason that we don't need to deform the $\mathcal{A}$-sheaves in N as *sheaves*, but can do this over the affine patches on X, ignoring the glueing.

Remark 2.4 The reader may wonder why we do not call $\mathfrak{X}_{\mathcal{A}}$ a non-commutative algebraic *scheme*. The reason is that if $\mathcal{A}$ is commutative, $\mathfrak{X}_{\mathcal{A}}$ is much bigger than $\mathbf{Spec}\,\mathcal{A}$. Hence, the scheme structure of $\mathbf{Spec}\,\mathcal{A}$ is just a small part of the $\mathfrak{X}_{\mathcal{A}}$.

We say that a ring $\mathfrak{o}$ is an *arithmetic ring*, if it is an order in a number field K. Normally, these rings will be the ring of integers $\mathfrak{o}_K$ in K.

Definition 2.5 If $\mathcal{A}$ is a finite-rank $\mathcal{O}_X$-algebra, where X is of finite type over an arithmetic ring, $\mathfrak{X}_{\mathcal{A}}$ is called a *non-commutative arithmetic space*.

It is important to note, however, that since the objects $\mathfrak{O}$ and $\mathfrak{O}_n$ are deformation-theoretic (i.e., local) constructions, they can only be defined fibre-wise. It is in general not possible to glue these objects over the whole base $\mathfrak{o}$.

Definition 2.6 The space $\mathfrak{X}_{\mathcal{A}}$ is *regular at* M if $\operatorname{Ext}^2_{\mathcal{A}}(\mathcal{M}_i, \mathcal{M}_j) = 0$ for all $\mathcal{M}_i, \mathcal{M}_j \in \mathsf{M}$; $\mathfrak{X}_{\mathcal{A}}$ is *regular* if it is regular at all families M.

It is worth pointing out that the above is a deformation-theoretic definition of regularity. Hence a point is regular if all deformations of that point are unobstructed. This is a strong condition. There are other, weaker, notions of non-commutative regularity (e.g., Auslander regularity) that we will encounter later (but won't define formally).

Finally, the following is important enough to warrant its own remark.

Remark 2.5 Let $\mathcal{A}$ be a non-commutative algebra over an affine S-scheme X. Then, taking the projective closure $f : X \hookrightarrow \mathbb{P}^n_S$, we can push-forward $\mathcal{A}$ to an algebra $f_*\mathcal{A}$ on $\operatorname{im} f$. This can be useful when considering non-commutative algebras that are finite over their centre since we then can use projective techniques (properness in particular) in the study of $\mathcal{A}$.

3 Point modules

In this section, we let Λ be a Dedekind domain and R a commutative Λ-algebra. We denote by A an R-algebra, typically non-commutative.

It turns out that it is surprisingly difficult to define points for non-commutative spaces over non-algebraically closed fields, in a way generalising the commutative situation naturally.

Let X be a scheme over a base S, which we assume is affine. In addition, let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra of finite rank. This implies that $\mathcal{A}$ is finite over its centre, and is thus a PI-algebra. Clearly, $\mathcal{O}_X \subseteq \mathfrak{Z}(\mathcal{A})$.

Put $Y := \mathbf{Spec}\,\mathfrak{Z}(\mathcal{A})$. Every ideal $\mathfrak{A} \subset \mathcal{A}$ has non-trivial intersection $\mathfrak{a} = \mathfrak{A} \cap \mathfrak{Z}(\mathcal{A})$ with the centre. Furthermore, if $\mathfrak{A}$ is prime (or maximal), then so is $\mathfrak{a}$ (see [1, III.1.1], for instance). Conversely, given $\mathfrak{a} \in \mathbf{Spec}\,\mathfrak{Z}(\mathcal{A})$, there are at most finitely many prime ideals $\mathfrak{A} \subset \mathcal{A}$ over $\mathfrak{a}$ (i.e., such that $\mathfrak{a} = \mathfrak{A} \cap \mathfrak{Z}(\mathcal{A})$). The intersection of ideals in $\mathcal{A}$ with the centre can be viewed as defining a finite, surjective, morphism

$$\Psi : \ \mathfrak{X}^{\mathrm{p}}_{\mathcal{A}} \to Y, \tag{2}$$

where $\mathfrak{X}_{\mathscr{A}}^{\mathrm{p}}$ is the subset of $\mathfrak{X}_{\mathscr{A}}$ consisting of modules having prime annihilators. In fact, Ψ exists even over the complement $\mathfrak{X}_{\mathscr{A}} \setminus \mathfrak{X}_{\mathscr{A}}^{\mathrm{p}}$, by identifying modules with their annihilator ideals. However, over the complement, saying that Ψ is finite is not a meaningful assertion. Nevertheless, we will say that Ψ is "finite over $\mathfrak{X}_{\mathscr{A}}$", when in reality we should say "finite over $\mathfrak{X}_{\mathscr{A}}^{\mathrm{p}}$".

The locus in Y where the extension of $\mathfrak{p}$ to $\mathscr{A}$ is a unique prime, is called the *Azumaya locus*, $\mathtt{azu}(\mathscr{A})$, of $\mathscr{A}$. This is a Zariski open subset of Spec $\mathfrak{Z}(\mathscr{A})$, and the complement is the support of a Cartier divisor (e.g., [10, III.2.5]), called the *ramification locus*, $\mathtt{ram}(\mathscr{A})$.

For the reader's convenience we list, in the remark below, the ring-theoretic properties that are, explicitly or implicitly, used in the definition of points. Everything can be found in standard texts on non-commutative algebra. We recommend in particular the excellent [21, Chapter 15].

Remark 3.1 Let R be a commutative *domain* over a Dedekind domain Λ and A an R-algebra. In addition, M is an A-module, of finite rank n as an R-module.

(i) For any R-representation $\varrho : A_{/R} \to \mathrm{End}_R(M)$, we have $\ker \varrho = \mathrm{Ann}(M)$.
(ii) An ideal $\mathfrak{a} \subset A$ is *primitive* if $\mathfrak{a} = \mathrm{Ann}(M)$, where M is a *simple* A-module. Notice that $\mathfrak{a}$ is two-sided.
(iii) A ring R is *primitive* if $R = A/\mathfrak{a}$, where $\mathfrak{a} \subset A$ is a primitive ideal.
(iv) Primitive ideals are prime ideals. Consequently, primitive rings are prime rings.
(v) Maximal ideals are primitive.
(vi) As a consequence of the above points we have

$$A/\ker \varrho = A/\,\mathrm{Ann}(M) \subseteq \mathrm{End}_R(M) \simeq M_n(R),$$

and since Λ is a Dedekind domain, $M_n(R)$ is a prime ring. Therefore, $A/\ker \varrho$ is also prime, implying that $\ker \varrho = \mathrm{Ann}(M)$ is a prime ideal.
(vii) Primitive ideals in artinian rings are maximal. But note that, unless R is artinian, $M_n(R)$ is not an artinian ring.
(viii) The centre of a simple algebra is a field.
(ix) If S is an artinian ring, then $S/J(S)$ is semi-simple, where $J(S)$ denotes the Jacobson radical of S. Hence, if S is in addition prime, S is semi-simple since $J(S)$ is nilpotent in an artinian ring.

Hopefully any uncertain claims below are covered by these points.

We assume from now on that A is an R-algebra finite over its centre. A *point* on $\mathfrak{X}_A$ is an R-representation $\boldsymbol{\varrho} : A_{/R} \to \mathrm{End}_R(M)$ such that $(\ker \boldsymbol{\varrho})|_{\mathfrak{Z}(A)}$ is a prime ideal.

Note that we, at this point, make no assumption on $\boldsymbol{\varrho}$ being simple or the rank of M being finite. We will identify $\boldsymbol{\varrho}$, $\ker \boldsymbol{\varrho}$, and the module M, referring to all these as *the* point $\boldsymbol{\varrho}$. We also note that, if ι is the inclusion $\iota : \mathfrak{Z}(A) \to A$, then

$$\ker(\boldsymbol{\varrho} \circ \iota) = (\ker \boldsymbol{\varrho})|_{\mathfrak{Z}(A)},$$

and

$$A/\ker(\boldsymbol{\varrho} \circ \iota) = A \otimes_{\mathfrak{Z}(A)} \frac{\mathfrak{Z}(A)}{\ker(\boldsymbol{\varrho} \circ \iota)},$$

i.e., $A/\ker(\boldsymbol{\varrho} \circ \iota)$ is the fibre of A over $\ker(\boldsymbol{\varrho} \circ \iota)$.

Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\} \subset \mathrm{Spec}\, A$ be the prime ideals in A such that $\ker(\boldsymbol{\varrho} \circ \iota) \subseteq \mathfrak{p}_i$, for all $1 \leq i \leq s$. Since A is finite over its centre, there are only finitely many such primes (see

[1, Prop. III.1.1]). The incomparability theorem for algebras that are finite over their centres (see [18, Thm. 13.8.14]), implies that $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ when $i \neq j$. We have

$$\ker(\boldsymbol{\varrho} \circ \iota) \subseteq \bigcap_{i=1}^{s} \mathfrak{p}_i, \quad \text{and} \quad \ker(\boldsymbol{\varrho} \circ \iota) = \Big(\bigcap_{i=1}^{s} \mathfrak{p}_i\Big)\Big|_{\mathfrak{Z}(A)},$$

and so natural maps

$$A/\ker(\boldsymbol{\varrho} \circ \iota) \to A\Big/\bigcap_{i=1}^{s} \mathfrak{p}_i \;\to\; \prod_{i=1}^{s} A/\mathfrak{p}_i.$$

If all the $\mathfrak{p}_i$ are maximal ideals the last map is an isomorphism given by the Chinese remainder theorem.

Put $\mathfrak{Z}(A)_i := \mathfrak{Z}(A)/(\mathfrak{Z}(A) \cap \mathfrak{p}_i)$ and note that $\mathfrak{Z}(A)_i = \mathfrak{Z}(A/\mathfrak{p}_i)$. In addition, we put $k := \Lambda/(\mathfrak{p}_i \cap \Lambda)$ (this is the same field for all i) and $k_i := R/(R \cap \mathfrak{p}_i)$. Let ϱ_i be the induced morphism $\varrho_i : A \to A/\mathfrak{p}_i$, and put

$$\text{Ét}(\boldsymbol{\varrho}) := \mathfrak{Z}(A)_1 \times \mathfrak{Z}(A)_2 \times \cdots \times \mathfrak{Z}(A)_s.$$

From now on we assume that the rank of M over R is finite.

Definition 3.1 Let $\boldsymbol{\varrho} : A \to \text{End}_R(M)$ be a point on $\mathcal{X}_A$.

(a) Then $\boldsymbol{\varrho}$ is a *closed étale point* if the R-algebras $\bar{\varrho}_i := A/\mathfrak{p}_i$ are all simple rings (equivalently, the $\mathfrak{p}_i$ are maximal). The algebras $\bar{\varrho}_i$ (or equivalently the $\mathfrak{p}_i$) are the *underlying points* of $\boldsymbol{\varrho}$. We denote by

$$P_{\boldsymbol{\varrho}} := \big\{\bar{\varrho}_i = A/\mathfrak{p}_i \;\mid\; 1 \leq i \leq s\big\},$$

the set of underlying points.

(b) The fields $K_i := \mathfrak{Z}(\bar{\varrho}_i)$ are the *residue fields* of $\boldsymbol{\varrho}$. The k-algebra $\text{Ét}(\boldsymbol{\varrho})$ is the *étale residue ring* of $\boldsymbol{\varrho}$. Each K_i is a finite extension of k_i, so the algebra $\text{Ét}(\boldsymbol{\varrho})$ is étale.

If $s = 1$, $\boldsymbol{\varrho}$ is simply a *closed point*, which we write ϱ (in non-boldface).

Example 3.1 Let k be a field and A the k-algebra $A := k\langle x, y\rangle/(xy + yx)$. We note first that $\mathfrak{Z}(A) = k[x^2, y^2]$. The maximal ideal in $\mathfrak{Z}(A)$ given by $\mathfrak{p} := (x^2 - a, y^2 - b)$, for $a, b \in k^\times \setminus (k^\times)^2$, defines a point via the surjection $A \twoheadrightarrow A/\mathfrak{p}$. Now, $A/\mathfrak{p}$ is the quaternion algebra

$$\mathbf{Q}_{a,b} : \quad x^2 = a, \quad y^2 = b, \quad xy = -xy$$

and hence simple. Therefore, $\mathfrak{p}$ defines a closed point on $\mathcal{X}_A$ with residue field k.

Étale "quaternionic points" can be defined by combining different prime ideals (i.e., choosing different a, b). For instance, the ideal

$$\bar{\mathfrak{p}} := \bigcap_{(a,b)\in I} \mathfrak{p}_{(a,b)}, \tag{3}$$

for (a, b) in some finite index set I, defines an étale point $\boldsymbol{\varrho} : A \twoheadrightarrow A/\bar{\mathfrak{p}}$ with underlying point $P_{\boldsymbol{\varrho}} = \{A/\mathfrak{p}_{(a,b)} \mid (a, b) \in I\}$ and residue ring $\text{Ét}(\boldsymbol{\varrho}) = \prod_I k$. Indeed, since $\bar{\mathfrak{p}} = \bigcap_{(a,b)\in I} \mathfrak{p}_{(a,b)}$ we have the surjection

$$A/\bar{\mathfrak{p}} \twoheadrightarrow \prod_{(a,b)\in I} A/\mathfrak{p}_{(a,b)}.$$

In fact, this is actually an isomorphism by the Chinese remainder theorem, since the $\mathfrak{p}_{(a,b)}$ are maximal ideals.

Next, we define rational points on non-commutative algebraic spaces. The definition can certainly be made more general, but writing this out would obscure its intended purpose in the present context. In order to motivate the definition, let's first look at an example.

Example 3.2 Let $T = \prod_{i=1}^{s} T_i/\mathfrak{m}_i$ be a semi-simple ring, where T_i are rings and $\mathfrak{m}_i \subset T_i$ maximal ideals. In addition, let $\boldsymbol{\xi} : A \to T$ be a ring map. We can decompose $\boldsymbol{\xi}$ into maps $\xi_i : A \to T_i/\mathfrak{m}_i$. Put $\mathfrak{p}_i := \xi_i^{-1}(\mathfrak{m}_i) = \ker \xi_i$. We have

$$\ker(\boldsymbol{\xi} \circ \iota) \subseteq \ker(\xi_i \circ \iota) \subseteq \mathfrak{p}_i$$

for all i and so

$$\ker(\boldsymbol{\xi} \circ \iota) \subseteq \bigcap_{i=1}^{s} \ker(\xi_i \circ \iota) \subseteq \bigcap_{i=1}^{s} \mathfrak{p}_i,$$

from which it follows that we have the following sequence of maps

$$A\Big/\ker(\boldsymbol{\xi} \circ \iota) \longrightarrow A\Big/\bigcap_{i=1}^{s} \ker(\xi_i \circ \iota) \longrightarrow A\Big/\bigcap_{i=1}^{s} \mathfrak{p}_i \longrightarrow \prod_{i=1}^{s} A/\mathfrak{p}_i.$$

In other words, when T is semi-simple, the map $\boldsymbol{\xi} : A \to T$ comes with a natural induced map

$$A\Big/\ker(\boldsymbol{\xi} \circ \iota) \to \prod_{i=1}^{s} A/\mathfrak{p}_i.$$

Definition 3.2 Let A and T be R-algebras, with A finite over its centre, ι the inclusion $\iota : \mathfrak{Z}(A) \to A$, and $\boldsymbol{\xi} : A \to T$ an R-algebra morphism.

(a) The morphism $\boldsymbol{\xi}$ is an *étale T-rational point* on $\mathfrak{X}_A$ if there is a map

$$A/\ker(\boldsymbol{\xi} \circ \iota) \xrightarrow{(\alpha_i)} \prod_{i=1}^{s} (A/\mathfrak{p}_i)^{n_i}, \quad 1 \leq s < \infty,$$

where $n_i \geq 1$ and the $\mathfrak{p}_i$ are the prime ideals in A such that $\ker(\boldsymbol{\xi} \circ \iota) \subset \mathfrak{p}_i$ for all $1 \leq i \leq s$. Some of the α_i might be zero.
Note that [1, Prop. III.1.1] ensures that $s < \infty$. In fact, $1 \leq s \leq \mathrm{rk}_{\mathfrak{Z}}(A)$.

(b) The *underlying T-rational étale point* of $\boldsymbol{\xi}$ is the algebra

$$E_T := A/\ker \boldsymbol{\xi} \hookrightarrow T$$

and the *underlying étale point* is the set of algebras

$$E_{\boldsymbol{\xi}} := \left\{\bar{\xi}_i := A/\mathfrak{p}_i\right\}.$$

The underlying étale point is *closed* if $E_{\boldsymbol{\xi}}$ is a set of simple algebras, equivalently, the $\mathfrak{p}_i$ are maximal.

(c) $\boldsymbol{\xi}$ is a *geometric étale point* if $\mathfrak{Z}(\bar{\xi}_i) = k^{\mathrm{al}}$ for all i for some algebraically closed field k^{al}.

We denote the étale T-rational points on $\mathfrak{X}_A$ by $\mathfrak{X}_A^{\text{ét}}(T)$. Obviously, if $s = 1$, we simply say that the point is T-rational, omitting "étale", and write $\mathfrak{X}_A(T)$ instead of $\mathfrak{X}_A^{\text{ét}}(T)$.

The simplest example is $\boldsymbol{\xi} : A \to A/\mathfrak{p}$ for some prime $\mathfrak{p} \subset A$. Now $\ker(\boldsymbol{\xi} \circ \iota) = \mathfrak{p} \cap \mathfrak{Z}(A)$ and so we have a natural map $A/\ker(\boldsymbol{\xi} \circ \iota) \to A/\mathfrak{p}$. Hence $\boldsymbol{\xi}$ defines an $A/\mathfrak{p}$-rational point.

Remark 3.2 Note the difference between "underlying étale T-*rational* point" and "underlying étale point". Note that this makes sense even in the case where A is commutative. The reader should spell out the geometric meaning of this for him/herself.

Remark 3.3 Since $\ker(\boldsymbol{\xi} \circ \iota) \subseteq \bigcap_{i=1}^{s} \mathfrak{p}_i$, we see that we automatically have morphisms

$$A\big/\ker(\boldsymbol{\xi} \circ \iota) \to A\Big/\bigcap_{i=1}^{s} \mathfrak{p}_i \to \prod_{i=1}^{s} (A/\mathfrak{p}_i)^{n_i}.$$

Therefore, demanding the existence of the map (α_i) is superfluous. We choose to include it in the definition for the purpose of clarity.

Remark 3.4 For notational simplicity we will most often only write out the case where all the n_i are equal to 1. In other words, we ignore specifying possible multiplicities in the notation.

Example 3.3 Let A be a commutative k-algebra and K/k a field extension. A k-algebra morphism $\xi : A \to K$ defines a K-rational point on $\operatorname{Spec} A$. Since K is a field, $\ker \xi$ is maximal ideal and so $K' := A/\ker \xi$ is a field (hence a prime ring) and K/K' is a field extension. The underlying K-rational point is $K' = A/\ker \xi$.

Let E be an étale algebra over k, $E = \prod K_i$, and $\boldsymbol{\xi} : A \to E$ a k-algebra morphism:

$$\boldsymbol{\xi} : A \xrightarrow{(\xi_i)} \prod_{i=1}^{s} K_i, \quad \xi_i : A \to K_i.$$

Then $\ker \boldsymbol{\xi} = \bigcap_{i=1}^{s} \ker \xi_i$, and so

$$A/\ker \boldsymbol{\xi} = A/\ker \xi_1 \times A/\ker \xi_2 \times \cdots \times A/\ker \xi_s,$$

where each $A/\ker \xi_i \subset K_i$. The underlying point is the set

$$E_{\boldsymbol{\xi}} = \left\{A/\ker \xi_i \;\mid\; 1 \le i \le s\right\}.$$

Hence an étale point in a commutative scheme, is simply a union of K_i-points on the scheme.

Example 3.4 Let L be a field and let $T = \operatorname{End}_L(V)$, where V is a finite-dimensional L-vector space. Because L is a field, T is a simple ring since $\operatorname{End}_L(V) \simeq M_n(L)$ for $n = \dim_L V$. Hence the kernel of $\boldsymbol{\xi} : A \to \operatorname{End}_L(V)$ is a maximal ideal $\mathfrak{p}$ such that $\ker(\boldsymbol{\xi} \circ \iota) \subseteq \mathfrak{p}$. Therefore, we have a map $A/\ker(\boldsymbol{\xi} \circ \iota) \to A/\mathfrak{p}$ and so $\boldsymbol{\xi}$ is a $\operatorname{End}_L(V)$-rational point. The underlying point is $A/\ker \boldsymbol{\xi} = A/\mathfrak{p}$.

The following example is important as it illustrates a general principle of lifting rational points on the centre, to rational points on $\mathfrak{X}_A$.

Example 3.5 Let $A := k\langle x, y\rangle/(xy + yx)$ be the algebra from example 3.1. Let L be a field and consider an L-rational point $\xi_{\mathfrak{Z}} : \mathfrak{Z}(A) \to L$ on the centre $\mathfrak{Z}(A) = k[x^2, y^2]$. The kernel $\mathfrak{a} := \ker \xi_{\mathfrak{Z}}$ is a maximal ideal and $K := k[x^2, y^2]/\mathfrak{a}$ is a subfield of L. Let us assume first that $\mathfrak{a} = (x^2 - a, y^2 - b)$, with $ab \neq 0$. This implies that $\mathfrak{a} \in \mathtt{azu}(A)$. Then the composition

$$\begin{tikzcd} A \arrow[r] \arrow[rrr, bend left, "\xi"] & A/\mathfrak{a} \arrow[r, equal] & A \otimes_{k[x^2,y^2]} K \arrow[r, hook] & A \otimes_{k[x^2,y^2]} L, \end{tikzcd}$$

defines an $A \otimes_{k[x^2,y^2]} L$-rational point on $\mathfrak{X}_A$, with underlying L-rational point $A \otimes_{k[x^2,y^2]} K$. Because of its construction, i.e., by lifting L-rational points on $\mathfrak{Z}(A)$ to A, it is reasonable to call such an ξ an "L-rational point on $\mathfrak{X}_A$". Observe that $A \otimes_{k[x^2,y^2]} K$ is a quaternion algebra over K.

On the other hand, if $ab = 0$, then $\mathfrak{a} \in \mathtt{ram}(A)$. Let's assume that $x^2 = 0$ (i.e., $a = 0$). Put

$$M := A/\mathfrak{a} = K \cdot \boldsymbol{e},$$

on which x and y act as, $x\boldsymbol{e} = s\boldsymbol{e}$ and $y\boldsymbol{e} = t\boldsymbol{e}$, for some $s, t \in K$. Because $x^2 = 0$, we clearly must have $s = 0$. Since $y^2 = b$ and $y^2\boldsymbol{e} = t^2\boldsymbol{e}$, we need to have $b = t^2$. From this we find that there are two maximal ideals in A over $\mathfrak{a}$, namely, $\mathfrak{p}_+ := (x, y - t)$ and $\mathfrak{p}_- := (x, y + t)$. Clearly, $\mathfrak{a} = \mathfrak{p}_+ \cap \mathfrak{p}_-$, and so the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rrr, bend left, "\alpha_1\times\alpha_2"] & A \otimes_{k[x^2,y^2]} K \arrow[r, "\simeq"] \arrow[d, equal] & A/\mathfrak{p}_+ \times A/\mathfrak{p}_- \arrow[r, equal] & \frac{A}{(x,y-t)} \times \frac{A}{(x,y+t)}\ , \\ & A/\mathfrak{a} & & \end{tikzcd}$$

with $K = k[x^2, y^2]/\mathfrak{a}$ as above, defines an étale L-rational point on $\mathfrak{X}_A$ with underlying L-rational (or $A \otimes_{k[x^2,y^2]} L$-rational) point $A \otimes_{k[x^2,y^2]} K$ and underlying closed point

$$E_\xi = \left\{A/\mathfrak{p}_+,\ A/\mathfrak{p}_-\right\}.$$

Notice the different roles that K and L play. Of course, K and L can be the same field.

Remark 3.5 There is a bijective correspondence

$$\left\{T\text{-rational points, } \xi : A \to T\right\} \quad \longleftrightarrow \quad \left\{(A/\ker\xi, j) \mid j : A/\ker\xi \hookrightarrow T\right\}.$$

Notice that T becomes a $\mathfrak{Z}(A/\ker\xi)$-algebra via j.

For any extension $T \subset T'$ the group $\mathrm{Aut}_T(T')$ acts on $\mathfrak{X}_A(T')$ as

$$(A/\ker\xi, j)^\sigma = (A/\ker\xi, \sigma \circ j)$$

for all $\sigma \in \mathrm{Aut}_T(T')$.

Example 3.6 Let A and S be R-algebras, where R is a commutative domain, and let $\xi : A \to S$ be an S-point on $\mathfrak{X}_A$. For any ring extension R' of R the base extension $\xi_{R'} := \xi \otimes R'$ defines an $S \otimes_R R'$-rational point.

In particular, if $S = \mathrm{End}_R(M)$ we have that

$$\xi_{R'} = \xi \otimes R' : A \to \mathrm{End}_R(M) \otimes_R R' = \mathrm{End}_{R'}(M \otimes_R R')$$

is a R'-rational point. The point $\xi_{R'}$ is geometric if R is a field and $R' = R^{\mathrm{al}}$.

Taking inspiration from the previous example, we can re-interpret deformations of representations in terms of rational points.

Example 3.7 Assume for simplicity that A is a prime ring. Let

$$\mathsf{N} := \{N_1, N_2, \ldots, N_n\}$$

be a family of A-modules of finite dimension over k and let $\varrho_i : A \to \mathrm{End}_k(N_i)$ be the corresponding structure morphisms. Since the N_i are finite-dimensional, the endomorphism rings $\mathrm{End}_k(N_i)$ are prime rings implying that $\ker \rho_i$ are all prime ideals.

Form the ring morphism ϱ by

$$\varrho : A \xrightarrow{(\varrho_i)} \bigoplus_{i=1}^{n} \mathrm{End}_k(N_i).$$

The kernel is

$$\ker \varrho = \bigcap_{i=1}^{n} \ker \varrho_i = \prod_{i=1}^{n} A/\ker \varrho_i.$$

If $\bigcap_{i=1}^{n} \ker \varrho_i = (0)$ we define $\ker \varrho$ to be $\prod A/\ker \varrho_i$. This ϱ then defines an étale point.

Recall that $\mathrm{End}_k(\mathsf{N})$ is the matrix ring $(\mathrm{Hom}_k(N_i, N_j))_{ij}$. Let $\Lambda = (\Lambda_{ij})_{ij}$ be an object in $(\mathrm{art}_k)_r$. Then the lifting of ϱ to Λ is

$$\varrho_\Lambda : A \to \mathrm{End}_k(\mathsf{N}) \otimes_k \Lambda = \mathrm{End}_\Lambda(\mathsf{N} \otimes_k \Lambda) = (\mathrm{Hom}_k(N_i, N_j) \otimes_k \Lambda_{ij}).$$

Note that $\mathrm{End}_\Lambda(\mathsf{N} \otimes_k \Lambda)$ is an artinian ring so $A/\ker \varrho_\Lambda$ is a prime ring. Hence, ϱ_Λ is an étale Λ-rational point (recall definition 3.2(d)) on $\mathfrak{X}_A$, reducing to the étale point ϱ via the radical of Λ.

Taking the pro-completion in $(\widehat{\mathrm{art}_k})_r$, we can construct $\mathrm{End}_\Lambda(\mathsf{N} \otimes_k \hat{\Lambda})$-points. In particular, the pro-representing hull defines an $\mathrm{End}_\Lambda(\mathsf{N} \otimes_k \hat{H})$-rational point on $\mathfrak{X}_A$. We refer to Sect. 2.2 for the details on these constructions.

4 Spaces that are finite over their centres

4.1 Points on spaces that are finite over their centres

Let Λ be a commutative base ring and let X be an S-scheme, where $S = \mathrm{Spec}\,\Lambda$. Denote by $\mathscr{A}$ a finite-rank $\mathcal{O}_X$-algebra and put $Y := \mathbf{Spec}\,\mathfrak{Z}(\mathscr{A})$. Note that this defines a finite morphism $\varphi : Y \to X$. Without loss of generality, we can work over affine patches. So assume that $X = \mathrm{Spec}\,R$, where R is a Λ-algebra, and that A is a finite-rank R-algebra.

If $\mathfrak{p}$ is a closed point on $\mathrm{Spec}\,\mathfrak{Z}(A)$ we put $k := \Lambda/\mathfrak{p} \cap \Lambda$. Let $\varrho : A \to \mathrm{End}_\Lambda(M)$ be a *closed* point in $\mathfrak{X}_A$ and put $\mathfrak{p} := \ker(\varrho \circ \iota)$, where $\iota : \mathfrak{Z}(A) \to A$ is the inclusion. Suppose that $\mathfrak{p} \in \mathtt{azu}(A) \subseteq Y$. Then $\Psi^{-1}(\mathfrak{p})$ is the unique maximal ideal above $\mathfrak{p}$. Conversely, if $\mathfrak{p} \in \mathtt{azu}(A)$, then the action of A on $A/\Psi^{-1}(\mathfrak{p})$, defines a closed point in $\mathfrak{X}_A$.

Since $A/\Psi^{-1}(\mathfrak{p})$ is simple, $\mathfrak{Z}(\varrho) = \mathfrak{Z}(A/\mathfrak{p})$ is a field, and Hilbert's Nullstellensatz gives us the finite field extensions

$$k \subseteq R/\mathfrak{p} \cap R \subseteq \mathfrak{Z}(A)/\mathfrak{p} \subseteq \mathfrak{Z}(\varrho).$$

In fact, $\mathfrak{Z}(\varrho) = \mathfrak{Z}(A)/\mathfrak{p}$, and so $A/\Psi^{-1}(\mathfrak{p})$ is a central simple algebra over $\mathfrak{Z}(A)/\mathfrak{p}$.

Suppose now that $\mathfrak{p} = \ker(\varrho \circ \iota) \in \mathtt{ram}(A)$. Then $\Psi^{-1}(\mathfrak{p}) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$, for $s \geq 1$. The set $\Psi^{-1}(\mathfrak{p})$ consists of all primes above $\mathfrak{p}$. By assumption ϱ is a *closed* point so these are by hypothesis maximal ideals. Note that s can still be 1, even if $\mathfrak{p} \in \mathtt{ram}(A)$. Therefore ϱ is closed étale point with underlying points

$$P_\varrho = \left\{\bar{\varrho}_i = A/\mathfrak{p}_i \mid 1 \leq i \leq s\right\}$$

and residue class fields $\mathfrak{Z}(\bar{\varrho}_i)$. In fact, $\mathfrak{Z}(\bar{\varrho}_i) = \mathfrak{Z}(\bar{\varrho}_j)$ for all $1 \leq i, j \leq s$.

4.2 Lifting central rational points

Let T be a Λ-algebra and $\xi_{\mathfrak{Z}} : \mathfrak{Z}(A) \to T$ a T-rational point on Spec $\mathfrak{Z}(A)$. We do not assume that T is commutative, but we do assume that it is prime.

Put $\mathfrak{a} := \ker \xi_{\mathfrak{Z}}$ as above. Since T is prime and $\mathfrak{Z}(A)/\mathfrak{a} \hookrightarrow T$, the quotient $S := \mathfrak{Z}(A)/\mathfrak{a}$ must be a domain, and so $\mathfrak{a}$ is a prime ideal. We thus have a map $\boldsymbol{\xi}$ defined by the composition

$$\begin{tikzcd} A \arrow[r] \arrow[rrr, bend left, "\boldsymbol{\xi}"] & A/\mathfrak{a} \arrow[r, equal] & A \otimes_{\mathfrak{Z}(A)} S \arrow[r, hook] & A \otimes_{\mathfrak{Z}(A)} T \end{tikzcd}$$

since $S \hookrightarrow T$. Note that $\mathfrak{a} = \ker(\boldsymbol{\xi} \circ \iota) = \ker \boldsymbol{\xi}$.

Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the prime ideals in A lying over $\mathfrak{a}$. Then $\mathfrak{a} \subseteq \bigcap_{i=1}^s \mathfrak{p}_i$ and hence there is a map

$$A/\mathfrak{a} \xrightarrow{(\alpha_i)} \prod_{i=1}^{s} A/\mathfrak{p}_i,$$

defined by the natural projections $\alpha_i : A/\mathfrak{a} \twoheadrightarrow A/\mathfrak{p}_i$. Therefore, the composition map $\boldsymbol{\xi}$ above is an $A \otimes_{\mathfrak{Z}(A)} T$-rational étale point with underlying étale point $A \otimes_{\mathfrak{Z}(A)} S = A/\mathfrak{a}$ and closed étale point

$$E_{\boldsymbol{\xi}} = \{A/\mathfrak{p}_1, \ldots, A/\mathfrak{p}_s\}.$$

The étale residue class ring of $\boldsymbol{\xi}$ is $\text{Ét}(\boldsymbol{\xi}) = \prod_{i=1}^s K_i$, where $K_i := \mathfrak{Z}(A/\mathfrak{p}_i) = \mathfrak{Z}(\xi_i)$.

Remark 4.1 If $\mathfrak{a}$ is *not* prime, the situation is more complicated, but can be resolved using the minimal prime ideals over $\mathfrak{a}$. Notice that this corresponds to looking at the irreducible components of the scheme Spec $\mathfrak{Z}(A)/\mathfrak{a}$. The situation can be even more complicated due to the fact that $\mathfrak{a}$ need not be reduced, and so the components comes with multiplicities. We avoid these complications by assuming that T is prime (and so $\mathfrak{a}$ prime).

Remark 4.2 Clearly, any étale T-rational point $\boldsymbol{\xi}$ on $\mathfrak{X}_A$ restricts to an étale T-rational point on the central subscheme Y and we get inclusions

$$\mathfrak{Z}(A)\big/(\ker \boldsymbol{\xi} \cap \mathfrak{Z}(A)) \longrightarrow A/\ker \boldsymbol{\xi} \longrightarrow T.$$

Notice that, unless $\ker \boldsymbol{\xi}$ is maximal, $\mathfrak{Z}(A)\big/(\ker \boldsymbol{\xi} \cap \mathfrak{Z}(A))$ is not necessarily a field.

Remark 4.3 Since the scheme morphism $\varphi : Y \to X$ is finite, it is clear that we can lift points on the base scheme X to points on $\mathfrak{X}_A$. However, the extension of points on X depends on the relation between $\varphi(\mathtt{azu}(A))$, $\varphi(\mathtt{ram}(A))$ and the branch divisor of φ itself, and to disentangle this dependence in the general case, is quite cumbersome.

4.3 Rational points and Brauer classes

For K a field (or, more generally, a commutative ring), we denote by $\mathrm{Br}(K)$ the Brauer group of K and $\mathrm{Br}(K)[n]$ its n-torsion part. Put $A_i := A/\mathfrak{p}_i$.

We denote by T an artinian, prime, Λ-algebra, and let $\boldsymbol{\xi} \in \mathfrak{X}_A^{\text{ét}}(T)$. Hence, there is a map

$$A/\ker(\boldsymbol{\xi} \circ \iota) \xrightarrow{(\alpha_i)} \prod_{i=1}^{s} A_i = \prod_{i=1}^{s} A/\mathfrak{p}_i,$$

where the $\mathfrak{p}_i$ are the prime ideals over $\mathfrak{a} := \ker(\boldsymbol{\xi} \circ \iota)$. Since T is a prime ring, the centres $\mathfrak{Z}(A/\mathfrak{p}_i)$ are domains.

That T is prime implies that $\ker \boldsymbol{\xi}$ and $\mathfrak{a}$ are both prime ideals. In addition, since T is by assumption artinian, so are the A_i. By [13, Prop. 10.16] primeness of each A_i implies that there are no non-zero nilpotent ideals in the A_i, and in particular the Jacobson radical is therefore zero in each. Now, theorem 4.14 in [13] implies that each A_i is a semi-simple ring. Therefore it is sufficient to assume that each A_i is simple, since otherwise s just becomes larger (we get more simple components). Hence, we can assume that the $\mathfrak{p}_i$ are maximal.

Then, the A_i being simple of finite rank, these are central simple algebras over $K_i := \mathfrak{Z}(A_i)$. As such, the A_i determines classes in $\mathrm{Br}(K_i)$. In fact, these classes lie in the torsion parts $\mathrm{Br}(K_i)[d_i]$, with $d_i := \dim_{K_i}(A_i)$. If $\ker(\boldsymbol{\xi} \circ \iota) \subset \mathtt{azu}(A)$, $s = 1$. Taking the compositum K' of all the k_i, we get central simple algebras $A_i \otimes_{K_i} K$, defining classes $\mathrm{Br}(K')$. However we note that K' might be a splitting field for some (or all) A_i, and so some (or all) of the induced classes might be trivial in $\mathrm{Br}(K')$.

Let L be a field, and $\xi_{\mathfrak{Z}} : \mathfrak{Z}(A) \to L$ an L-rational point with kernel $\mathfrak{a}$. Note that $\mathfrak{a}$ is a maximal ideal. As in Sect. 4.2, we can lift of $\xi_{\mathfrak{Z}}$ to A through $K := \mathfrak{Z}(A)/\mathfrak{a}$, by

$$\boldsymbol{\xi} : A \xrightarrow{\xi_K} A \otimes_{\mathfrak{Z}(A)} K = A/\mathfrak{a} \longrightarrow A \otimes_{\mathfrak{Z}(A)} L.$$

This determines an $A \otimes_{\mathfrak{Z}(A)} L$-rational point on $\mathfrak{X}_A$. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the maximal ideals of A over $\mathfrak{a}$; then $\mathfrak{a} \subseteq \bigcap_{i=1}^{s} \mathfrak{p}_i$. Hence, we find a map $A/\mathfrak{a} \to \prod_{i=1}^{s} A_i$, where each A_i is central simple over $K_i = \mathfrak{Z}(A_i)$. Therefore, we get elements

$$[A_i] \in \mathrm{Br}(K_i)[d_i], \quad d_i = \dim_{K_i}(A_i).$$

If $\mathfrak{a} \in \mathtt{azu}(A)$, we have $s = 1$.

Therefore, we get an association

$$E_{\boldsymbol{\xi}} \longrightarrow \Big\{[A_i] \in \mathrm{Br}(K_i)[d_i] \;\Big|\; 1 \leq i \leq s, \;\; d_i = \dim_{K_i}(A_i)\Big\}.$$

Clearly, it would be hard to define a natural association in the opposite direction since different representatives in a $[A_i]$, would give different $\boldsymbol{\xi}$.

We summarise the above discussions in the following proposition.

Proposition 4.1 *Let $\boldsymbol{\xi} \in \mathfrak{X}_A^{ét}(T)$, where T is an artinian, prime, Λ-algebra, and let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the prime ideals of A over $\mathfrak{a} := \ker(\boldsymbol{\xi} \circ \iota)$. In addition, let L be a field. We note that the $\mathfrak{Z}(A_i)$ are not necessarily fields.*

(a) Put $k_i := \mathfrak{Z}(A_i)_{(0)}$, where the subscript (0) denotes localisation at the generic point. Then $\boldsymbol{\xi}$ defines (not necessarily distinct or non-trivial) classes in $\mathrm{Br}(k_i)[d_i]$, where $d_i := \dim_{k_i}(A_i)$.

(b) Let $K' := \prod_{i=1}^{s} k_i$ be the compositum of the fields k_i. Then $\boldsymbol{\xi}$ defines (possibly trivial) classes in $\mathrm{Br}(K')$, corresponding to the underlying K'-points $A \otimes_{k_i} K'$.

(c) To each central L-rational point $\xi_{\mathfrak{Z}} : \mathfrak{Z}(A) \to L$, we can associate a family of Brauer classes via the association

$$E_{\boldsymbol{\xi}} \longrightarrow \Big\{[A_i] \in \mathrm{Br}(K_i)[d_i] \;\Big|\; 1 \leq i \leq s, \;\; d_i = \dim_{K_i}(A_i)\Big\},$$

as above.

4.4 The tangent structure of lifts of central S-points

Let $\iota : \mathfrak{Z}(A) \to A$ denote the inclusion morphism.

Recall that a *Jacobson ring* is a ring in which every prime ideal is the intersection of primitive ideals. The most important examples are fields, Dedekind domains with infinitely many prime ideals and affine algebras over Jacobson rings. Local rings of (Krull) dimension ≥ 1, are not Jacobson rings.

Let A be a k-algebra where k is a field. Assume also that A is Jacobson. Then A *satisfies the Nullstellensatz* if, for every *simple* A-module M, $\operatorname{End}_A(M)$ is algebraic over k. If A is Jacobson over a ring Λ, then A satisfies the Nullstellensatz if $A/\mathfrak{p}$ satisfies the Nullstellensatz over $\Lambda/\mathfrak{p} \cap \Lambda$ for $\mathfrak{p} \in \operatorname{Spec} \Lambda$ such that $\Lambda \cap \mathfrak{p}$ is maximal.

Proposition 4.2 *Let A be a polynomial identity algebra over a Jacobson ring $\Lambda \subseteq \mathfrak{Z}(A)$. Then*

(a) if M is a simple A-module, $\operatorname{Ann}(M)$ *is a maximal ideal and M is finite-dimensional over $\Lambda/\operatorname{Ann}(M) \cap \Lambda$;*
(b) A satisfies the Nullstellensatz.

Proof This follows immediately from theorem 13.10.4 in [18]. □

We begin by stating a version of Müller's theorem concerning the tangential structure of the fibres of Ψ.

Theorem 4.1 (Müller's theorem) *Let A be an affine PI-algebra over a field k and let M and N be simple finite-dimensional A-modules. Then*

$$\operatorname{Ext}^1_A(M, N) \neq \emptyset \quad \Longleftrightarrow \quad \operatorname{Ann}(M) \cap \mathfrak{Z}(A) = \operatorname{Ann}(N) \cap \mathfrak{Z}(A).$$

(The annihilators are left *ideals.)*

Proof This is a reformulation of Müller's theorem as stated in [1, Thm. III.9.2] using [1, Lem. I.16.2]. □

Proposition 4.2(a) shows that $\operatorname{Ann}(M)$ and $\operatorname{Ann}(N)$ are maximal ideals since M and N are simple. In addition, the intersection between a primitive ideal (in particular any maximal ideal) and the centre is a (central) maximal ideal (see [1, Prop. III.1.1.5]). Hence, putting $\mathfrak{m} := \operatorname{Ann}(M) \cap \mathfrak{Z}(A) = \operatorname{Ann}(N) \cap \mathfrak{Z}(A)$, we can rephrase the equivalence as

$$\operatorname{Ext}^1_A(M, N) \neq \emptyset \quad \Longleftrightarrow \quad \mathfrak{m} \in \mathtt{ram}(A).$$

In addition, observe that Müller's theorem is indeed a statement concerning the fibres of the contraction map Ψ.

Let $\boldsymbol{\xi} \in \mathfrak{X}_A(S)$ be an étale S-rational point on $\mathfrak{X}_A$, with S a prime artinian k-algebra, where k is a field. By Remark 3.1 (ix), we can thus assume that S is semi-simple. Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\} \subset A$ be the maximal ideals over $\ker(\boldsymbol{\xi} \circ \iota)$. Then we have a morphism

$$A/\ker(\boldsymbol{\xi} \circ \iota) \to \prod_{i=1}^{s} A/\mathfrak{p}_i = \prod_{i=1}^{s} \bar{\xi}_i,$$

and where $\xi_i : A \to \operatorname{End}_k(A/\mathfrak{m}_i)$ are the associated closed points (with underlying points $\bar{\xi}_i = A/\mathfrak{m}_i$; recall that these are *algebras*, while the ξ_i represent the $A/\mathfrak{m}_i$ as *(left) A-modules*). Since the $\mathfrak{p}_i$ are maximal (hence primitive) ideals, so is $\ker(\boldsymbol{\xi} \circ \iota)$.

Therefore, Müller's theorem can be interpreted as a statement concerning rational points:

Theorem 4.2 (The rational Müller theorem) *Let A be an affine PI-algebra over a field k and let $\boldsymbol{\xi} \in \mathfrak{X}_A(S)$, where S is a prime artinian (semi-simple) ring. Then*

$$\operatorname{Ext}^1_A(\xi_i, \xi_j) \neq \emptyset \iff \ker \xi_i \cap \mathfrak{Z}(A) = \ker \xi_j \cap \mathfrak{Z}(A),$$

where we have used the notation $\operatorname{Ext}^1_A(\xi_i, \xi_j)$ for $\operatorname{Ext}^1_A(A/\mathfrak{m}_i, A/\mathfrak{m}_j)$. In addition, using the tangent space notation from Sect. 2.1, we note

$$\boldsymbol{T}_{\boldsymbol{\xi}} = \big(\operatorname{Ext}^1_A(\xi_i, \xi_j)_{ij}\big).$$

Despite Müller's theorem, there is nothing saying that $\{\bar{\xi}_i\} \subset \mathtt{ram}(A)$ in an étale point. It is à priori possible that $\xi_j \notin \mathtt{ram}(A)$, for some j.

Proposition 4.3 *Let $\mathfrak{m} \in \mathtt{azu}(A)$. Then*

$$\operatorname{Ext}^1_A(A/\mathfrak{m}, A/\mathfrak{m}) \simeq \operatorname{Ext}^1_{\mathfrak{Z}(A)}(k(\mathfrak{m}), k(\mathfrak{m})).$$

In other words, the central simple algebra $A/\mathfrak{m}$ have the same deformation theory as $\mathfrak{Z}(A)$ over $\mathtt{azu}(A)$.

Proof We will use the following change of base theorem for Ext^1. Let $R \to S$ be a ring morphism, M_R a left R-module and M_S a left S-module. Then

$$\operatorname{Ext}^1_S(M_R \otimes_R S, M_S) \simeq \operatorname{Ext}^1_R(M_R, M_S).$$

Take $R = \mathfrak{Z}(A)$, $S = A$, $M_R = \mathfrak{Z}(A)/\mathfrak{m}$ and $M_S = A/\mathfrak{m}$. Then

$$\begin{tikzcd}
\operatorname{Ext}^1_A(\mathfrak{Z}(A)/\mathfrak{m} \otimes_{\mathfrak{Z}(A)} A, A/\mathfrak{m}) \arrow[r, "\simeq"] \arrow[d, equal] & \operatorname{Ext}^1_{\mathfrak{Z}(A)}(\mathfrak{Z}(A)/\mathfrak{m}, A/\mathfrak{m}) \\
\operatorname{Ext}^1_A(A/\mathfrak{m}, A/\mathfrak{m}) &
\end{tikzcd}$$

Representing Ext in terms of Hochschild cohomology, we have

$$\operatorname{Ext}^1_A(M, N) \simeq \operatorname{Der}_k(A, \operatorname{Hom}_k(M, N))/\operatorname{Ad},$$

where Ad is the group of inner derivations. Thus there is a surjection

$$\operatorname{Ext}^1_{\mathfrak{Z}(A)}(\mathfrak{Z}(A)/\mathfrak{m}, A/\mathfrak{m}) \twoheadrightarrow \operatorname{Der}_k(\mathfrak{Z}(A), \operatorname{Hom}_k(\mathfrak{Z}(A)/\mathfrak{m}, A/\mathfrak{m}))$$

with kernel Ad.

Now, any $\phi : \mathfrak{Z}(A)/\mathfrak{m} \to A/\mathfrak{m}$ gives a morphism $\mathfrak{Z}(A)/\mathfrak{m} \to \mathfrak{Z}(A)/\mathfrak{m}$ by restriction. Conversely, any $\psi : \mathfrak{Z}(A)/\mathfrak{m} \to \mathfrak{Z}(A)/\mathfrak{m}$ certainly gives a morphism $\mathfrak{Z}(A)/\mathfrak{m} \to A/\mathfrak{m}$ by composing with the injection. Hence,

$$\operatorname{Hom}_k(\mathfrak{Z}(A)/\mathfrak{m}, A/\mathfrak{m}) = \operatorname{Hom}_k(\mathfrak{Z}(A)/\mathfrak{m}, \mathfrak{Z}(A)/\mathfrak{m})$$

and so

$$\begin{aligned}
\operatorname{Der}_k(\mathfrak{Z}(A), \operatorname{Hom}_k(\mathfrak{Z}(A)/\mathfrak{m}, A/\mathfrak{m})) &= \operatorname{Der}_k(\mathfrak{Z}(A), \operatorname{Hom}_k(\mathfrak{Z}(A)/\mathfrak{m}, \mathfrak{Z}(A)/\mathfrak{m})) \\
&= \operatorname{Der}_k(\mathfrak{Z}(A), \operatorname{Hom}_k(k(\mathfrak{m}), k(\mathfrak{m}))),
\end{aligned}$$

which proves the proposition. □

Let $\mathtt{smooth}(A)$ denote the smooth locus of $\mathfrak{Z}(A)$.

Theorem 4.3 *Let $\boldsymbol{\xi} : A \to S$ be an étale S-rational point on $\mathfrak{X}_A$.*

(a) *Suppose* $\ker \boldsymbol{\xi} \cap \mathfrak{Z}(A) \subset \mathtt{smooth}(A) \subseteq \mathtt{azu}(A)$. *Then the deformations of* $\{\xi_i\}$ *are unobstructed and the versal family ("formal S-rational point") is*

$$\hat{\boldsymbol{\xi}} = (\hat{\xi}_i) : A \to \bigoplus_{i=1}^{s} \operatorname{End}_k(\xi_i) \otimes_k k\langle\langle t_1, t_2, \dots, t_n \rangle\rangle,$$

where

$$n = \dim_k(\operatorname{Ext}^1_A(\xi_i, \xi_i)) = \dim_k(\operatorname{Ext}^1_{\mathfrak{Z}(A)}(k(\mathfrak{m}), k(\mathfrak{m}))) = \dim_k(T_{\mathfrak{m}} Y),$$

where $T_{\mathfrak{m}} Y$ *denotes the tangent space at* $\mathfrak{m} \in Y = \operatorname{Spec} \mathfrak{Z}(A)$.
Observe that, since $\mathfrak{m} \in \mathtt{smooth}(A)$, *the dimension of the* Ext-*spaces is independent on the* ξ_i.

(b) *If A is prime, of finite global dimension (for instance, if A is Auslander-regular) and finite as a module over its centre, then*

$$\mathtt{sing}(A) \cap \mathtt{azu}(A) = \emptyset.$$

Hence, the claims of (a) applies to all of $\mathtt{azu}(A)$ *(and any singularities must lie in* $\mathtt{ram}(A)$*).*

(c) *Assume* $\ker \boldsymbol{\xi} \cap \mathfrak{Z}(A) \subset \mathtt{ram}(A)$. *Then the versal family ("formal S-rational point") is*

$$\hat{\boldsymbol{\xi}} = (\hat{\xi}_i) : A \to \Big(\operatorname{Hom}_k(\xi_i, \xi_j) \otimes_k \hat{H}_{ij} \Big),$$

where

$$\hat{H}_{ii} \simeq k\langle\langle t_1, t_2, \dots, t_{n_i} \rangle\rangle \big/ (f_1, f_2, \dots, f_{m_i}),$$

and

$$n_i = \dim_k(\operatorname{Ext}^1_A(\xi_i, \xi_i)) \quad \text{and} \quad m_i = \dim_k(\operatorname{Ext}^2_A(\xi_i, \xi_i)).$$

The elements off the diagonal are more complicated to express in general, and are not algebras but ideals.

Note that, in the smooth part of the ramification locus, the deformations are unobstructed (i.e., $f_1 = f_2 = \cdots = f_{m_i} = 0$ *for all i). The case where* ξ *intersects both* $\mathtt{azu}(A)$ *and* $\mathtt{ram}(A)$ *clearly splits into two disjoint cases.*

Proof The claim concerning the unobstructedness in (a) follows from proposition 4.3 and the fact that smooth points deforms without obstruction. The versal family is given a direct consequence of the definition of unobstructed deformations of representations as given in Sect. 2.2. Proposition 4.3 also implies the claim concerning the dimensions. Part (b) follows from [1, Lemma III.1.8]. The claims in (c) is also follows from the discussion in Sect. 2.2.□

The above theorem gives a complete description of the ring objects $\mathfrak{O}$ in the case when A is PI-algebra.

4.5 Central topology

The above indicates that there is a close relationship between the geometry of $\mathfrak{X}_{\mathscr{A}}$ and the geometry of $\mathbf{Spec}\, \mathfrak{Z}(\mathscr{A})$. In the proposition below we will freely use the identification in (1) to identify modules with their corresponding annihilator ideals. We don't need to assume that the annihilator ideals are prime.

Proposition 4.4 *Let* $\mathscr{A}$ *be a PI-algebra over* $\mathcal{O}_X$, *with centre* $\mathfrak{Z}(\mathscr{A})$.

*(i) Suppose $D_{f'}$ is a distinguished open set on $\mathfrak{X}_{\mathscr{A}}$. Then $D_{f'} \cap$ **Spec** $\mathfrak{Z}(\mathscr{A})$ is a distinguished open set on* **Spec** *$\mathfrak{Z}(\mathscr{A})$.*

(ii) Conversely, suppose D_f is a distinguished open on **Spec** *$\mathfrak{Z}(\mathscr{A})$. Then $D_{f'}$, where $f = f'|_{\mathfrak{Z}(\mathscr{A})}$, is a distinguished open in $\mathfrak{X}_{\mathscr{A}}$.*

As a consequence, since the distinguished open sets are basis sets for the Zariski and Zariski–Jacobson topologies on **Spec** *$\mathfrak{Z}(\mathscr{A})$ and $\mathfrak{X}_{\mathscr{A}}$, respectively, the Zariski–Jacobson topology on $\mathfrak{X}_{\mathscr{A}}$ is compatible with the Zariski topology on* **Spec** *$\mathfrak{Z}(\mathscr{A})$.*

Proof Suppose $\mathfrak{a}' \in D_{f'}$. Then $f' \notin \mathfrak{a}'$ and so $f'|_{\mathfrak{Z}(\mathscr{A})} \notin \mathfrak{a}' \cap \mathfrak{Z}(\mathscr{A})$, in other words, $\mathfrak{a}' \cap \mathfrak{Z}(\mathscr{A}) \in D_{f'|_{\mathfrak{Z}(\mathscr{A})}}$. Conversely, suppose that $\mathfrak{a} \in D_f \subset$ **Spec** $\mathfrak{Z}(\mathscr{A})$ and take some $f' \in \mathscr{A}$ such that $f'|_{\mathfrak{Z}(\mathscr{A})} = f$. The extension of $\mathfrak{a}$ to $\mathscr{A}$ can be split into a number of ideals $\{\mathfrak{a}'_i \subset \mathscr{A}\}$. Suppose $f \in \mathfrak{a}'_i$ for some i. Then $f'|_{\mathfrak{Z}(\mathscr{A})} \in \mathfrak{a}'_i \cap \mathfrak{Z}(\mathscr{A}) \iff f \in \mathfrak{a}$, a contradiction. Hence $f' \notin \mathfrak{a}'_i$ and so the extension of D_f to $\mathfrak{X}_{\mathscr{A}}$ is a distinguished open set. □

If we view $\mathscr{A}$ as a sheaf of algebras over its centre **Spec** $\mathfrak{Z}(\mathscr{A})$, we can use central localisation, i.e., localisation of $\mathscr{A}$ at multiplicatively closed sets in its centre (such a localisation is always well-defined), and find

$$\mathscr{A}(D_f) = \mathscr{A}_f \quad \text{and} \quad \mathfrak{X}_{\mathscr{A}}(D_f) = \mathfrak{X}_{\mathscr{A}_f}.$$

Observe that these two statements are different. The first one says that the sheaf $\mathscr{A}$ is equal to $\mathscr{A}_f$ over $D_f \subseteq$ **Spec** $\mathfrak{Z}(\mathscr{A})$ on the centre, while the other says that the open set D_f, viewed as a distinguished open on $\mathfrak{X}_{\mathscr{A}}$, is equal to $\mathfrak{X}_{\mathscr{A}_f}$.

However, the proposition says more: we can localise on $\mathscr{A}$ directly by simply restricting the distinguished open sets $D_{f'}$ of $\mathfrak{X}_{\mathscr{A}}$ to distinguished opens D_f on **Spec** $\mathfrak{Z}(\mathscr{A})$.

There are a number of constructions that follow from proposition 4.4. We list the most obvious and important ones below.

4.6 Sheaves and invertible modules

Proposition 4.4 allows us to define sheaves on $\mathfrak{X}_{\mathscr{A}}$ with the Zariski–Jacobson topology, as sheaves on **Spec** $\mathfrak{Z}(\mathscr{A})$, defined by the restrictions $D_f \mapsto D_{f|_{\mathfrak{Z}(\mathscr{A})}}$.

Definition 4.1 An *$\mathscr{A}$-module* on $\mathfrak{X}_{\mathscr{A}}$ is a sheaf $\mathscr{F}$ on **Spec** $\mathfrak{Z}(\mathscr{A})$ such that each $\mathscr{F}(D_f) := \mathscr{F}(D_{f|_{\mathfrak{Z}(\mathscr{A})}})$ is an $\mathscr{A}_{f|_{\mathfrak{Z}(\mathscr{A})}}$-module. Notice that this implies that $\mathscr{F}$ is automatically an $\mathscr{O}_{\mathfrak{Z}(\mathscr{A})}$-module (we write $\mathscr{O}_{\mathfrak{Z}(\mathscr{A})}$ instead of $\mathscr{O}_{\mathbf{Spec}\,\mathfrak{Z}(\mathscr{A})}$ to simplify notation).

An algebra A is *generically free* if every finitely generated A-module M is *centrally* locally free, in the sense that there is an $f \in \mathfrak{Z}(A)$ such that $M_f := M \otimes_A A_f$ is free as an $A_f := A \otimes_{\mathfrak{Z}(A)} \mathfrak{Z}(A)_f$-module.

It is a fact that every affine PI-algebra over a Jacobson ring is generically free (see [20, Theorem 6.3.3]). Therefore, for affine PI-algebras $\mathscr{A}$ over Jacobson schemes, finitely generated $\mathscr{A}$-modules are locally free. Hence,

Proposition 4.5 *Every finitely generated module over $\mathfrak{X}_{\mathscr{A}}$, where $\mathfrak{Z}(\mathscr{A})$ is a Jacobson ring, is locally free as an $\mathscr{A}$-module.*

For simplicity of notation, we will temporarily work with affine algebras over a commutative ring Λ. An *invertible* A-module is a finitely generated (so, if A is a Jacobson ring, locally free by proposition 4.5), projective, A-bimodule such that

$$A \simeq \mathrm{End}_A({}_AM) \quad \text{and} \quad A \simeq \mathrm{End}_A(M_A),$$

where we denote the left action of A on M by ${}_AM$ and similarly the right action.

The set of isomorphism classes of all such form a group, denoted $\mathrm{Pic}(A)$, under tensor products (see [6]):

$$[M]\cdot[N] := [M\otimes_A N].$$

This is well-defined since M and N are A-bimodules. The inverse of the invertible module M is

$$M^{-1} := \mathrm{Hom}_A(M, A).$$

Let R be a commutative subring of A and let M be an A-bimodule. If $Mr = rM$ for all $r\in R$, we say that M is defined *over* R. Notice that this need not be the case in general since R can act differently from the left and from the right.

We denote the set of invertible A-modules over R by $\mathrm{Pic}_R(A)$ (or $\mathrm{Pic}_R(\mathcal{X}_A)$), and by

$$\mathrm{Pic}_R^{\mathrm{lf}}(A) := \mathrm{Pic}_R^{\mathrm{lf}}(\mathcal{X}_A) := \Big\{M\in \mathrm{Pic}_R(A)\;\Big|\; M \text{ locally free over } R\Big\}.$$

If $R=\mathfrak{Z}(A)$ we put $\mathrm{Pic}_{\mathfrak{Z}}(A) := \mathrm{Pic}_{\mathfrak{Z}(A)}(A)$. By proposition 4.5, if A is a Jacobson ring, $\mathrm{Pic}_R^{\mathrm{lf}}(A) = \mathrm{Pic}_R(A)$.

Recall that Ψ is the morphism defined by contracting ideals from $\mathscr{A}$ to $\mathfrak{Z}(\mathscr{A})$. For a locally free sheaf $\mathscr{F}$ of rank f on $\mathrm{Spec}\,\mathfrak{Z}(\mathscr{A})$, denote by $\det(\mathscr{F})$ the top exterior product

$$\det(\mathscr{F}) = \bigwedge^f \mathscr{F} = \mathscr{F}\wedge\mathscr{F}\wedge\cdots\wedge\mathscr{F} \qquad (f \text{ times}).$$

Definition 4.2 An invertible sheaf $\mathscr{L}\in \mathrm{Pic}_{\mathfrak{Z}}^{\mathrm{lf}}(\mathscr{A})$ is *very ample* if

$$\det(\Psi_*\mathscr{L}) \simeq \mathscr{N}|_{\mathbf{Spec}\,\mathfrak{Z}(\mathscr{A})},$$

for $\mathscr{N}$ a very ample invertible sheaf on $\mathbb{P}(\mathfrak{Z}(\mathscr{A}))$. In other words, there is an embedding $\alpha:\mathbb{P}(\mathfrak{Z}(\mathscr{A}))\hookrightarrow \mathbb{P}^m$, for some $m\geq 1$, such that $\mathscr{N}\simeq \alpha^*\mathscr{O}(1)$. Put

$$\mathrm{Pic}_{\mathfrak{Z}}^{\mathrm{va}}(\mathscr{A}) := \mathrm{Pic}_{\mathfrak{Z}}^{\mathrm{va}}(\mathcal{X}_{\mathscr{A}}) := \Big\{\mathscr{L}\in \mathrm{Pic}_{\mathfrak{Z}}^{\mathrm{lf}}(\mathscr{A})\;\Big|\;\mathscr{L} \text{ is very ample}\Big\}.$$

We also make the following definition:

Definition 4.3 The sheaf $\mathscr{C}\in \mathrm{Pic}_{\mathfrak{Z}}^{\mathrm{lf}}(\mathscr{A})$ is a *canonical sheaf* if

$$\det(\Psi_*\mathscr{C}) \simeq \mathscr{C}_{\mathfrak{Z}},$$

for $\mathscr{C}_{\mathfrak{Z}}$ a canonical sheaf on $\mathbb{P}(\mathfrak{Z}(\mathscr{A}))$.

We will not use this definition explicitly in what follows.

4.7 "Morphisms" between non-commutative spaces

Let $\mathscr{A}$ and $\mathscr{B}$ be two algebras over the same k-scheme X (where k is a field), and let

$$\psi:\;\; \mathscr{B}\to\mathscr{A}$$

be an algebra morphism. This defines a *set-theoretic* morphism

$$\psi^{\natural}:\;\; \mathcal{X}_{\mathscr{A}}\to\mathcal{X}_{\mathscr{B}}$$

by restriction of modules. A "morphism" (which we will call a "non-commutative morphism") between $\mathcal{X}_{\mathscr{A}}$ and $\mathcal{X}_{\mathscr{B}}$ should respect both the Zariski–Jacobson topology and map $\mathscr{O}_{\mathcal{X}_{\mathscr{B}}}$ to $\mathscr{O}_{\mathcal{X}_{\mathscr{A}}}$. We now show that this can be done in a formal sense.

Theorem 4.4 *Let ψ be as above and let*

$$\mathsf{N} := \left\{\mathscr{N}_1, \mathscr{N}_2, \ldots, \mathscr{N}_n\right\}$$

be a finite family of $\mathscr{A}$-modules (over X) such that $\dim_k(\mathscr{N}_i) < \infty$ *(cf. Remark 2.2). Then ψ induces a* non-commutative morphism

$$\psi^\natural : \ \mathfrak{X}_{\mathscr{A}} \to \mathfrak{X}_{\mathscr{B}}$$

defined by the composition

$$\mathscr{B} \xrightarrow{\psi} \mathscr{A} \xrightarrow{\varrho} \mathscr{E}nd_{\mathscr{O}_X}(\mathsf{N}). \tag{4}$$

Proof We work over affine patches. It will be clear that everything glues.

So, let $\boldsymbol{\varrho} = \oplus \varrho : A \to \mathrm{End}_k(\mathsf{N})$ be a family of A-modules. Then, clearly, $\boldsymbol{\varrho} \circ \psi : B \to \mathrm{End}_k(\mathsf{N})$ is a family of B-modules.

Deforming N as A-modules gives the versal family

$$\hat{\boldsymbol{\varrho}} : \ A \longrightarrow \left(\mathrm{Hom}_k(N_i, N_j) \otimes_k \hat{H}_{ij}\right)$$

inducing

$$\hat{\boldsymbol{\varrho}} \circ \psi : \ B \longrightarrow A \longrightarrow \left(\mathrm{Hom}_k(N_i, N_j) \otimes_k \hat{H}_{ij}\right).$$

Deforming $\boldsymbol{\varrho} \circ \psi$ directly gives a morphism

$$B \longrightarrow \left(\mathrm{Hom}_k(N_i^B, N_j^B) \otimes_k \hat{H}(\boldsymbol{\varrho} \circ \psi)_{ij}\right),$$

where N_i^B means viewing N_i as B-module via $\boldsymbol{\varrho} \circ \psi$. By versality, we get an induced morphism

$$\hat{\psi}_{ij} : \ \left(\mathrm{Hom}_k(N_i^B, N_j^B) \otimes_k \hat{H}(\boldsymbol{\varrho} \circ \psi)_{ij}\right) \longrightarrow \left(\mathrm{Hom}_k(N_i, N_j) \otimes_k \hat{H}_{ij}\right),$$

giving (by definition)

$$\hat{\psi}_{ij} : \ \left(\mathrm{Hom}_k(N_i^B, N_j^B) \otimes_k \hat{H}(\boldsymbol{\varrho} \circ \psi)_{ij}\right) \longrightarrow \hat{\mathscr{O}}_{\mathsf{N}}.$$

Putting

$$\hat{\mathscr{O}}_{\mathsf{N}^B} := \left(\mathrm{Hom}_k(N_i^B, N_j^B) \otimes_k \hat{H}(\boldsymbol{\varrho} \circ \psi)_{ij}\right)$$

we get

$$\hat{\psi}_{ij} : \ \hat{\mathscr{O}}_{\mathsf{N}^B} \longrightarrow \hat{\mathscr{O}}_{\mathsf{N}},$$

giving, formally, the desired local (algebra) morphism of the "structure sheaves" $\boldsymbol{\mathscr{O}}$.

As for the topology, this is essentially obvious. Let $f \in B$ and let $M \in D_f$ with structure morphism $\varrho : B \to \mathrm{End}_k(M)$. Hence, by definition, $\varrho(f)M \neq 0$. That M is in the image of $\psi^\natural$ set-theoretically means that there is a $\sigma : A \to \mathrm{End}_k(M)$ such that $\varrho = \sigma \circ \psi$. Then, if $\sigma(\psi(f))M = 0$, we would have that $\varrho(f)M = 0$. Hence, $(\psi^\natural)^{-1}(D_f) = D_{\psi(f)}$, completing the proof that $\psi^\natural$ is continuous for the Zariski–Jacobson topology. □

If X is an S-scheme where S is not the spectrum of a field, we get a *topological* map $\mathfrak{X}_{\mathscr{A}} \to \mathfrak{X}_{\mathscr{B}}$ from (4). The morphisms $\hat{\psi}$ between the $\mathscr{O}$-rings (as in the above proof) can only be constructed *fibre-by-fibre* over S.

4.8 Central subschemes and their non-commutative lifts

Let $Y := \mathbf{Spec}\,\mathfrak{Z}(\mathscr{A})$ and let $W \overset{j}{\hookrightarrow} Y$ be a closed subscheme. Since $\mathscr{A}$ is a locally free Y-algebra, we can restrict $\mathscr{A}$ to W via j. This means that $j^*\mathscr{A}$ is a locally free W-algebra with $\mathcal{O}_W \subseteq \mathfrak{Z}(j^*\mathscr{A})$.

Algebraically, we have

$$\begin{tikzcd} \mathscr{A} \arrow[r] & \mathscr{A} \otimes_{\mathfrak{Z}(\mathscr{A})} \big(\mathfrak{Z}(\mathscr{A})/\mathscr{I}\big) = \mathscr{A}/\langle \mathscr{I} \rangle \\ \mathfrak{Z}(\mathscr{A}) \arrow[u, hook] \arrow[r, dash] & \mathfrak{Z}(\mathscr{A})/\mathscr{I} \arrow[u, hook] \end{tikzcd}$$

and geometrically

$$\begin{tikzcd} \mathsf{Mod}\ j^*\mathscr{A} \arrow[r] \arrow[d, dotted] & \mathsf{Mod}\ \mathscr{A} \arrow[d, dotted] \\ W \arrow[r, hook] & Y, \end{tikzcd} \tag{5}$$

where the dotted arrows indicate that the morphisms are defined by restriction of ideals.

In fact we can extend this to include the $\mathfrak{O}$-rings.

Proposition 4.6 *With the notation as above, diagram (5) can be completed to the diagram*

$$\begin{tikzcd} \mathfrak{X}_{j^*\mathscr{A}} \arrow[r] \arrow[d, dotted] & \mathfrak{X}_{\mathscr{A}} \arrow[d, dotted] \\ W \arrow[r, hook] & Y. \end{tikzcd} \tag{6}$$

In this way $\mathfrak{X}_{j^*\mathscr{A}}$ defines a closed non-commutative subspace *of $\mathfrak{X}_{\mathscr{A}}$.*

Proof The compatibility between the topologies is obvious so we only need to prove that the $\mathfrak{O}$-ring on $\mathsf{Mod}\,A$ restricts to $\mathsf{Mod}\,j^*A$.

The argument is essentially the same deformation-theoretic argument used in the proof of Theorem 4.4. We work locally, so let $\varrho : A/I \to \mathrm{End}_k(\mathsf{N})$ be an étale point on $\mathsf{Mod}\,j^*A$. This gives the étale point $\mathrm{pr} \circ \varrho : A \to \mathrm{End}_k(\mathsf{N})$ on $\mathsf{Mod}\,A$, where $\mathrm{pr} : A \to A/I$ is the projection.

Deforming N as an A/I-module gives the versal family

$$\hat{\varrho}_{A/I} : \quad A/I \longrightarrow \Big(\mathrm{Hom}_k(N_i, N_j) \otimes_k \hat{H}(A/I)_{ij}\Big)$$

and as an A-module gives the family

$$\hat{\varrho} : \quad A \longrightarrow \Big(\mathrm{Hom}_k(N_i, N_j) \otimes_k \hat{H}_{ij}\Big)$$

By versality we get an algebra morphism

$$\Big(\mathrm{Hom}_k(N_i, N_j) \otimes_k \hat{H}_{ij}\Big) \longrightarrow \Big(\mathrm{Hom}_k(N_i, N_j) \otimes_k \hat{H}(A/I)_{ij}\Big)$$

which gives the desired restriction of $\mathfrak{O}$-rings from $\mathsf{Mod}\,A$ to $\mathsf{Mod}\,j^*A$. $\square$

In the case of PI-algebras, the interesting case is when $W \cap \mathtt{ram}(\mathscr{A}) \neq \emptyset$. Unfortunately, we have to leave this analysis for another time.

5 Non-commutative diophantine geometry

In this section we introduce three height functions: the first is a (classical) height function on the centre, the second is a height function that measures how complicated the rational point is as representations, and the third is a height function that is purely non-commutative in the sense that it also takes into account the tangent structure of the point.

5.1 Height functions

A height function on an algebraic variety $X_{/k}$ over a number field k (or function field for that matter) is a function

$$H_K:\ X_{/k}(K) \longrightarrow \mathbb{R}, \qquad k \subseteq K,$$

i.e., a function on (the coordinates of) K-rational points with values in $\mathbb{R}$. We will follow [8, Chapter B], for which we refer for more details. Put

$$\mathbb{R}(X) := \Big\{H : X(k^{\mathrm{al}}) \to \mathbb{R}\Big\}$$

and call elements in this set *height functions*. If we were to take a more serious and in-depth look at the subject, we should consider $\mathbb{R}(X)$ modulo bounded functions. However the above is more than sufficient for our purpose here.

The basic example is the following. Let X be the n-dimensional projective space $\mathbb{P}^n$ and let Σ_k be the set of valuations of k:

$$\Sigma_k = \Sigma_k^{\mathrm{f}} \cup \Sigma_k^{\infty},$$

where Σ_k^{f} is the set of non-archimedean (finite) valuations and Σ_k^{∞}, the set of archimedean (infinite) valuations. We denote the, to $v \in \Sigma_k$ associated normalised absolute value, $\|\cdot\|_v$.

Let $\boldsymbol{p} = (p_0 : p_1 : \cdots : p_m) \in \mathbb{P}^m(k)$ and define the *(Weil) height* of $\boldsymbol{p}$ to be

$$H_k(\boldsymbol{p}) := \prod_{v \in \Sigma_k} \max\Big\{\|p_0\|_v, \|p_1\|_v, \dots, \|p_m\|_v\Big\},$$

and the *logarithmic height* as

$$h_k(\boldsymbol{p}) := \log H_k(\boldsymbol{p}) = -\sum_{v \in \Sigma_k} n_v \min\Big\{v(p_0), v(p_1), \dots, v(p_m)\Big\},$$

where $n_v = [k_v/\mathbb{Q}_p]$ and $v \mid p$.

For $k \subseteq K$ a finite extension, [8, Lemma B.2.1(c)] gives

$$H_K(\boldsymbol{p}) = H_k(\boldsymbol{p})^{[K/k]}. \tag{7}$$

The following is therefore a natural definition: we define a *height sequence* to be a sequence $\{H_k \mid \mathbb{Q} \subseteq k\}$, parametrised by the field extensions of $\mathbb{Q}$, with the different H_k coherent in the sense that (7) holds if $k \subseteq K$.

We also define the *absolute height* of $\boldsymbol{p}$ to be

$$H(\boldsymbol{p}) := H_K(\boldsymbol{p})^{\frac{1}{[K/\mathbb{Q}]}}$$

where K is any field such that $\boldsymbol{p} \in \mathbb{P}^m(K)$. This definition is independent on K. Similarly we define the *absolute logarithmic height* to be

$$h(\boldsymbol{p}) := \log H(\boldsymbol{p}) = \frac{1}{[K/\mathbb{Q}]} h_K(\boldsymbol{p}).$$

5.2 Height functions on $\mathfrak{Z}(A)$

Now let $\mathbb{P}(\mathfrak{Z}(A)) := \mathbb{P}(\operatorname{Spec}\mathfrak{Z}(A))$ be the projective closure of $\operatorname{Spec}\mathfrak{Z}(A)$:

$$\operatorname{Proj}: \ \operatorname{Spec}\mathfrak{Z}(A) \hookrightarrow \mathbb{P}(\mathfrak{Z}(A)).$$

As $\mathbb{P}(\mathfrak{Z}(A))$ is a projective scheme there is a closed embedding

$$\alpha: \ \mathbb{P}(\mathfrak{Z}(A)) \hookrightarrow \mathbb{P}^m$$

for some m. Put $\varphi := \alpha \circ \operatorname{Proj}$ and let $\boldsymbol{p} \in (\operatorname{Spec}\mathfrak{Z}(A))(K)$ be a K-rational point. We then define the *height of* $\boldsymbol{p}$ *relative to* φ to be the function

$$H_\varphi(\boldsymbol{p}) := H(\varphi(\boldsymbol{p})),$$

where H is the absolute height function on $\mathbb{P}^m$. We also define the *logarithmic height relative to* φ as

$$h_\varphi(\boldsymbol{p}) := h(\varphi(\boldsymbol{p})).$$

5.3 The (naïve) central heights

In the context of height functions it is reasonable to assume that S is prime and artinian, so semi-simple by Remark 3.1 (ix). We make this assumption for the rest of this section.

Let $\boldsymbol{\xi} \in \mathfrak{X}_A(S)$ be an étale S-rational point with $\ker\boldsymbol{\xi} = \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_2$. Put $\bar{\mathfrak{m}}_i := \mathfrak{m}_i \cap \mathfrak{Z}(A)$, viewed as points in $\operatorname{Spec}\mathfrak{Z}(A)$.

We now make the following naïve definition.

Definition 5.1 Let $\boldsymbol{\xi} \in \mathfrak{X}_A(S)$ be an étale S-rational point. Then the *central height* of $\boldsymbol{\xi}$ relative to φ is the vector

$$H_\varphi^{\mathfrak{Z}}(\boldsymbol{\xi}) := \Big(H\big(\varphi(\bar{\mathfrak{m}}_1)\big), H\big(\varphi(\bar{\mathfrak{m}}_2)\big), \ldots, H\big(\varphi(\bar{\mathfrak{m}}_s)\big)\Big),$$

with its associated logarithmic counterpart

$$h_\varphi^{\mathfrak{Z}}(\boldsymbol{\xi}) := \Big(h\big(\varphi(\bar{\mathfrak{m}}_1)\big), h\big(\varphi(\bar{\mathfrak{m}}_2)\big), \ldots, h\big(\varphi(\bar{\mathfrak{m}}_s)\big)\Big),$$

We denote by $\mathbb{R}^{\mathfrak{Z}}(\mathfrak{X}_A)$ the set of all central heights on $\mathfrak{X}_A$. This set is clearly parametrised by the embeddings $\alpha : \mathbb{P}(\mathfrak{Z}(A)) \hookrightarrow \mathbb{P}^m$.

In other words, if $\Psi : \mathfrak{X}_A \to \operatorname{Spec}\mathfrak{Z}(A)$ is the morphism defined by contraction, we have

$$H_\varphi^{\mathfrak{Z}} = H_\varphi \circ \Psi = H \circ \varphi \circ \Psi = H \circ \alpha \circ \operatorname{Proj} \circ \Psi,$$

where α, Proj and φ are defined above. In fact, since the projective closure is canonical, this construction is only dependent on α. Hence, we write H_α for H_φ.

Recall that $\operatorname{Pic}_{\mathfrak{Z}}^{\mathrm{va}}(\mathfrak{X}_{\mathscr{A}})$ (see Definition 4.2) is the set of all very ample invertible sheaves on $\mathfrak{X}_A$. The following is a (slightly) non-commutative variant of Weil's Height Machine (see [8, Chapter B]).

Theorem 5.1 *Let $\mathscr{A}$ be a PI-algebra with centre $\mathfrak{Z}(\mathscr{A})$.*

(a) We have set-theoretic maps

$$\begin{tikzcd} \operatorname{Pic}_{\mathfrak{Z}}^{va}(\mathfrak{X}_{\mathscr{A}}) \arrow[r] & \operatorname{Pic}(\boldsymbol{Spec}\,\mathfrak{Z}(\mathscr{A}))^{\circledast} \arrow[r] & \mathbb{R}^{\mathfrak{Z}}(\mathfrak{X}_{\mathscr{A}}) \\ \mathscr{L} \arrow[r, mapsto] & \det\big((\Psi_*\mathscr{L})|_{\boldsymbol{Spec}\,\mathfrak{Z}(\mathscr{A})}\big) \arrow[r, mapsto] & H_\alpha^{\mathfrak{Z}}, \end{tikzcd}$$

where α is the embedding $\alpha : \mathbb{P}(\textbf{\textit{Spec}}\,\mathfrak{Z}(\mathscr{A})) \hookrightarrow \mathbb{P}^m$, associated with $\mathscr{L}$ via its determinant on $\textbf{\textit{Spec}}\,\mathfrak{Z}(\mathscr{A})$. The map $\circledast$ is the classical Weil Height Machine.

(b) *If $\mathscr{L}$ is not very ample, i.e., if $\det\big((\Psi_*\mathscr{L})|_{\textbf{\textit{Spec}}\,\mathfrak{Z}(\mathscr{A})}\big)$ is not very ample, we can find two very ample $\mathscr{L}_1$ and $\mathscr{L}_2$ on $\mathbb{P}(\textbf{\textit{Spec}}\,\mathfrak{Z}(\mathscr{A}))$ such that*

$$\det\big((\Psi_*\mathscr{L})|_{\textbf{\textit{Spec}}\,\mathfrak{Z}(\mathscr{A})}\big) \simeq \mathscr{L}_1 \otimes \mathscr{L}_2^{-1}.$$

Therefore we can define $H^{\mathfrak{Z}}_{\alpha} := H^{\mathfrak{Z}}_{\alpha_1} - H^{\mathfrak{Z}}_{\alpha_2}$, giving

$$\begin{array}{ccccc} \operatorname{Pic}(\mathfrak{X}_{\mathscr{A}}) & \longrightarrow & \operatorname{Pic}(\textbf{\textit{Spec}}\,\mathfrak{Z}(\mathscr{A})) & \overset{\circledast}{\longrightarrow} & \mathbb{R}^{\mathfrak{Z}}(\mathfrak{X}_{\mathscr{A}}) \\ \mathscr{L} & \longmapsto & \det\big((\phi_*\mathscr{L})|_{\textbf{\textit{Spec}}\,\mathfrak{Z}(\mathscr{A})}\big) & \longmapsto & H^{\mathfrak{Z}}_{\alpha}. \end{array}$$

It is important to observe that $(\mathscr{L}_1, \alpha_1)$ and $(\mathscr{L}_2, \alpha_2)$ are not unique. However, the height functions $H^{\mathfrak{Z}}_{\alpha}$ coming from different choices of $(\mathscr{L}_1, \alpha_1)$ and $(\mathscr{L}_2, \alpha_2)$, differ "only" up to bounded functions (see for example [8, Theorems B.3.2 and B.3.6]).

Proof Everything, except the map $\circledast$, follow directly from construction. For the construction of $\circledast$, see [8, Theorems B.3.2 and B.3.6]. That any invertible sheaf on a projective scheme can be written as (a multiplicative) difference of two very ample ones is well-known, but let's spell out an argument nevertheless. The twists $\mathcal{O}_X(n)$ are very ample for all n and if $\mathscr{L}$ is an invertible sheaf, $\mathscr{L} \otimes \mathcal{O}_X(n)$ is very ample for n sufficiently large by [7, Theorem II.5.17 and Exercise II.7.5(d)]. Hence $\mathscr{L} \simeq \mathscr{L} \otimes \mathcal{O}_X(n) \otimes \mathcal{O}_X(n)^{-1}$ □

Remark 5.1 It seems interesting to consider *ramification heights*, i.e., heights associated with the ramification locus. However, this is something that has to wait for another time.

5.4 Representation heights

The above was perhaps the most naive and obvious notion of height possible for PI-algebras. We will now construct a more "non-commutative version" that works for all finitely generated algebras.

Let $\boldsymbol{\xi} \in \mathfrak{X}_A(S)$ be an S-rational point. We will write out the construction for non-étale points. This implies that we assume that S is simple. The étale case, i.e., when S is semi-simple, will be obvious.

Put $\mathfrak{m} := \ker \boldsymbol{\xi}$, $\bar{\mathfrak{m}} := \mathfrak{m} \cap \mathfrak{Z}(A)$ and $K := \mathfrak{Z}(A/\mathfrak{m})$. Since S assumed to be simple, $\mathfrak{m}$, and hence also $\bar{\mathfrak{m}}$, is a maximal ideal. Note that $A/\mathfrak{m}$ is a finite-dimensional K-vector space as it is central simple over its centre (which in turn is a finite extension of k). The point $\boldsymbol{\xi}$ thus defines a representation $\xi : A \to \operatorname{End}_K(A/\mathfrak{m})$ (via the projection $A \twoheadrightarrow A/\mathfrak{m}$).

Let $\{x_1, x_2, \ldots, x_n\}$ be a set of generators for A. Since $A/\mathfrak{m}$ is simple as K-algebra, there is, by Wedderburn's theorem, a unique division algebra D with $\mathfrak{Z}(D) = K$, such that $A/\mathfrak{m} \simeq M_m(D)$, where m is also uniquely determined by $A/\mathfrak{m}$. Hence we can view the $\xi(x_i)$ as matrices with entries in D. These matrices are the *coordinates* of $\boldsymbol{\xi}$.

Let v be a henselian $\mathbb{R}$-valued valuation on K and K_v the completion of K with respect to v. For any finite extension $K_v \subset L$ the v extends uniquely to L. Therefore v extends uniquely to $D \otimes_K K_v$ (see [24, Thm. 1.4]) and we have a morphism

$$\beta : A/\mathfrak{m} \xrightarrow{\simeq} M_m(D) \to M_m(D \otimes_K K_v).$$

Denote by w_D the to $D_v := D \otimes_K K_v$ *uniquely* extended valuation of v and put $N_i := \beta(\xi(x_i))$. Applying the D-valuation w_D to all entries in N_i we get matrices $w_D(N_i) \in M_m(\mathbb{R})$.

Definition 5.2 Put $d_i := \det(w_D(N_i))$. Then we define the *logarithmic height* of $\boldsymbol{\xi}$ as.

$$h_K^{\mathtt{rep}}(\boldsymbol{\xi}) := -\sum_{v \in \Sigma_K} n_v \min\{d_1, d_2, \ldots, d_{m_i}\}, \quad n_v := [D \otimes_K K_v/k].$$

The corresponding absolute height is defined as $H_K^{\mathtt{rep}}(\boldsymbol{\xi}) := \exp(h_K^{\mathtt{rep}}(\boldsymbol{\xi}))$.

5.5 Non-commutative heights

Let $\boldsymbol{\xi} \in \mathfrak{X}_A(S)$ and let $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_s\}$ be the maximal ideals corresponding to $\ker \boldsymbol{\xi}$ (remember S is assumed to be semi-simple). Put, in addition, $\bar{\mathfrak{m}}_i := \mathfrak{m}_i \cap \mathfrak{Z}(A)$, and

$$\mathsf{P} := \Psi^{-1}(\Psi(\boldsymbol{\xi})) = \left\{\Psi^{-1}(\bar{\mathfrak{m}}_i) \;\middle|\; 1 \leq i \leq s\right\} = \left\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r\right\}, \quad r \geq s.$$

Notice that this set can include more points than the points in the decomposition of $\ker \boldsymbol{\xi}$, depending on whether some of the $\bar{\mathfrak{m}}_i \in \mathtt{ram}(A)$ or not.

The set P defines a new étale point:

$$\boldsymbol{\varrho} = (\varrho_i) : A \to \prod_{i=1}^{r} A/\mathfrak{p}_i$$

with and underlying points $\bar{\varrho}_i$. Put $(\mathbf{T}_{\mathsf{P}})_{ij} := \mathrm{Ext}^1_A(\varrho_i, \varrho_j)$ (cf. Sect. 2.1).

The tangent space graph $\Gamma_{\mathsf{P}} = \Gamma^1_{\mathsf{P}}$ then measures the noncommutativity of the point $\boldsymbol{\xi}$. Put $e_{ij} := \dim_k((\mathbf{T}_{\mathsf{P}})_{ij})$.

The adjacency matrix of Γ_{P} is

$$M_{\mathsf{P}} := \begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1r} \\ e_{21} & e_{22} & \cdots & e_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ e_{r1} & e_{r2} & \cdots & e_{rr} \end{pmatrix},$$

encoding the Γ_{P} in matrix form. Two graphs are isomorphic if their adjacency matrices are conjugate (similar). Hence, up to conjugacy (similarity), the ordering of the points $\boldsymbol{\xi}_i$ (and thus the $\mathfrak{p}_i$) is irrelevant. As a consequence, the set Λ of eigenvalues is an invariant of Γ_{P} and $\mathbf{T}_{\mathsf{P}}$.

Recall that $K = \mathfrak{Z}(A/\mathfrak{m})$. We then make the following definition.

Definition 5.3 The *non-commutative height* of $\boldsymbol{\xi}$ is

$$H_K^{\mathtt{nc}}(\boldsymbol{\xi}) := \prod_{\sigma: K \hookrightarrow K^{\mathrm{al}}} \max\left\{\|\sigma(\lambda_1)\|, \|\sigma(\lambda_2)\|, \ldots, \|\sigma(\lambda_t)\|\right\}.$$

The *total height* of $\boldsymbol{\xi}$ is the vector

$$H_{K,\alpha}^{\mathtt{tot}}(\boldsymbol{\xi}) := \left(H_\alpha^{\mathfrak{Z}}(\boldsymbol{\xi}), H_K^{\mathtt{rep}}(\boldsymbol{\xi}), H_K^{\mathtt{nc}}(\boldsymbol{\xi})\right) \in \mathbb{R}^2 \times \mathbb{C}.$$

Observe that this definition is dependent on the embedding $\alpha : \mathbb{P}(\mathfrak{Z}(A)) \hookrightarrow \mathbb{P}^n$. According to the Height Machine $H_\alpha^{\mathfrak{Z}}(\boldsymbol{\xi})$ can be given in terms of a very ample sheaf on the central scheme and a choice of α.

6 Arithmetic geometry of PI-algebras

6.1 Compactifying the base

Let k be a number field and $\mathfrak{o}$ an order in k (i.e., $\mathfrak{o}$ need not be integrally closed in K; we will later assume this though). Assume that X is a scheme of finite type over $\mathfrak{o}$.

6.1.1 Compactification of orders

Most of what will follow in this subsection can be found in [19, Chapter 3], although we will frame the discussion in terms of pseudo-divisors.

Let $\mathfrak{o}$ be an order in a number field k and let Σ be the set of infinite primes, i.e., the set of embeddings $k \hookrightarrow \mathbb{C}$. We compactify $\mathfrak{o}$ as

$$\bar{\mathfrak{o}} := \mathfrak{o} \times \Sigma.$$

Put $Y := \operatorname{Spec} \bar{\mathfrak{o}}$. Hence $\Sigma = Y(\mathbb{C})$. We will sometimes use the notation $Y^{\mathrm{f}} := \operatorname{Spec} \mathfrak{o}$ and $Y^{\infty} := \Sigma$.

A finitely generated $\mathfrak{o}$-module M extends to a module over $\bar{\mathfrak{o}}$ by extending M to

$$M_{\mathbb{C}} = M \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{\sigma \in \Sigma} M_\sigma = \bigoplus_{\sigma \in \Sigma} M \otimes_{\mathfrak{o},\sigma} \mathbb{C}, \qquad M_\sigma = M \otimes_{\mathfrak{o},\sigma} \mathbb{C},$$

where $M \otimes_{\mathfrak{o},\sigma} \mathbb{C}$ of course means that we view $\mathbb{C}$ as an $\mathfrak{o}$-module via $\sigma : \mathfrak{o} \hookrightarrow \mathbb{C}$. We put

$$\overline{M} := M \times M_{\mathbb{C}}.$$

Definition 6.1 Let $Y = \operatorname{Spec} \bar{\mathfrak{o}}$. Then an *Arakelov–Cartier divisor* on Y is a pair of pseudo-divisors

$$\left(\overline{\mathscr{L}}, \overline{Z}, \bar{s}\right) := \left(\mathscr{L}^{\mathrm{f}}, \mathscr{L}^{\infty}\right) := \Big((\mathscr{L}, Z, s), (\mathscr{L}_{\mathbb{C}}, Z_{\mathbb{C}}, s_{\mathbb{C}})\Big),$$

where $Z_{\mathbb{C}} \subseteq \Sigma$ and $s_{\mathbb{C}}$ a function $\mathscr{L}_{\mathbb{C}} \to \mathbb{C}^{\times}$, non-vanishing on $\Sigma \setminus Z_{\mathbb{C}}$. If $\overline{Z}$ and $\bar{s}$ are given or irrelevant we simply write $\overline{\mathscr{L}}$ for $\left(\overline{\mathscr{L}}, \overline{Z}, \bar{s}\right)$.

Notice that $\mathscr{L}_{\mathbb{C}}$ need not be the base change of $\mathscr{L}$ to $\mathbb{C}$.

Let D be a Cartier divisor on Y^{f}. Recall that a Cartier divisor on Y^{f} can be identified with the invertible ideals of $\mathfrak{o}$ (i.e., the finitely generated $\mathfrak{o}$-modules $I \subset k$ such that there is another finitely generated $I^{-1} \subset k$ with $I \otimes I^{-1} = \mathfrak{o}$).

Then D determines an Arakelov–Cartier divisor $\left(\mathscr{L}^{\mathrm{f}}, \mathscr{L}^{\infty}\right)$ with

$$\mathscr{L}^{\mathrm{f}} = \left(\mathscr{O}(D), |D|, s\right), \quad \text{and} \quad \mathscr{L}^{\infty} = \left(\mathscr{O}(D)_{\mathbb{C}}, \Sigma, 1_{\Sigma}\right).$$

To spell it out explicitly, let

$$D = \Big\{(U_i, f_i) \mid 1 \leq i \leq m, \;\; f_i \in k\Big\}$$

be a Cartier divisor on Y. The support of D is

$$|D| = \Big\{\mathfrak{p} \in \operatorname{Spec} \mathfrak{o} \mid f_i \notin \mathfrak{o}_{\mathfrak{p}}^{\times} \text{ for some } i\Big\}.$$

Then

$$\mathscr{O}(D) = \mathfrak{o} \cdot f_1^{-1} + \mathfrak{o} \cdot f_2^{-1} + \cdots + \mathfrak{o} \cdot f_m^{-1}, \qquad s = \prod_{i=1}^{m} f_i$$

(we write s multiplicatively) and

$$\mathcal{O}(D)_{\mathbb{C}} = \mathcal{O}(D) \otimes_{\mathbb{Z}} \mathbb{C} = \prod_{\sigma \in \Sigma} \mathcal{O}(D) \otimes_{\mathfrak{o},\sigma} \mathbb{C}, \quad \text{with} \quad 1_{\Sigma}(\sigma) = 1 \in \mathbb{C}^{\times}, \ \sigma \in \Sigma.$$

Obviously, by Σ we mean all embeddings of $k \hookrightarrow \mathbb{C}$ as before.

6.1.2 Compactification of algebras over arithmetic schemes

We compactify $X_{/\mathfrak{o}}$ to a scheme $\overline{X}_{/\bar{\mathfrak{o}}}$ by adding the complex points of X:

$$\overline{X} := \overline{X}_{/\bar{\mathfrak{o}}} := X_{/\mathfrak{o}} \times X^{\infty}, \quad \text{where} \quad X^{\infty} := \prod_{\sigma \in \Sigma} X \times_{\mathfrak{o},\sigma} \operatorname{Spec} \mathbb{C}.$$

We call $X := X_{/\mathfrak{o}}$ the *finite* part (we include the generic fibre in the finite part) and X^{∞} the *analytic* (or *infinite*) part of $\overline{X}_{/\bar{\mathfrak{o}}}$.

Let $\mathcal{A}$ be a finitely generated algebra over X, with structure morphism $f : \mathcal{O}_X \to \mathcal{A}$. We extend this to an analytic algebra as follows. Fix an embedding $\sigma : \mathfrak{o} \hookrightarrow \mathbb{C}$. Put

$$\mathcal{O}_X \otimes_{\mathfrak{o},\sigma} \mathbb{C} \xrightarrow{f \otimes_{\sigma} \mathbb{C}} \sigma^* \mathcal{A} := \mathcal{A} \otimes_{\mathfrak{o},\sigma} \mathbb{C}$$

and

$$\prod_{\sigma \in \Sigma} \left(\mathcal{O}_X \otimes_{\mathfrak{o},\sigma} \mathbb{C} \xrightarrow{f_{\sigma}} \sigma^* \mathcal{A} \right), \quad f_{\sigma} := f \otimes_{\sigma} \mathbb{C}.$$

We will normally work with one σ at a time for simplicity of notation. Finally we put

$$\overline{\mathcal{A}} := \mathcal{A} \times \prod_{\sigma \in \Sigma} \sigma^* \mathcal{A}.$$

We call $\overline{\mathcal{A}}$ the *compactification* of $\mathcal{A}$ over $\overline{X}$.

Let $\mathcal{M}$ be a left $\mathcal{A}$-module, finite over $\mathcal{O}_X$. We extend $\mathcal{M}$ to an $\overline{\mathcal{A}}$-module by

$$\overline{\mathcal{M}} := \mathcal{M} \times \prod_{\sigma \in \Sigma} \sigma^* \mathcal{M}, \quad \text{with} \quad \sigma^* \mathcal{M} := \mathcal{M} \otimes_{\mathfrak{o},\sigma} \mathbb{C}$$

Hence, $\sigma^* \mathcal{M}$ are sheaves over X^{∞} with an action by $\mathcal{A}$ and where $\mathfrak{o}$ acts via σ.

The category of all $\overline{\mathcal{A}}$-modules up to isomorphism is denoted $\mathsf{Mod}\, \overline{\mathcal{A}}$ and is the product category

$$\mathsf{Mod}\, \overline{\mathcal{A}} := \mathsf{Mod}\, \mathcal{A} \times \prod_{\sigma \in \Sigma} \mathsf{Mod}\, \sigma^* \mathcal{A}.$$

This meaning of this notation is hopefully clear.

Definition 6.2 We define $\overline{\mathfrak{X}}_{\mathcal{A}}$ as

$$\overline{\mathfrak{X}}_{\mathcal{A}} := (\mathfrak{X}^{\mathrm{f}}_{\mathcal{A}}, \mathfrak{X}^{\infty}_{\mathcal{A}}, \overline{\mathfrak{O}}) := \left(\mathsf{Mod}\, \mathcal{A} \times \prod_{\sigma \in \Sigma} \mathsf{Mod}\, \sigma^* \mathcal{A}, \overline{\mathfrak{O}} \right).$$

Clearly, $\overline{\mathfrak{O}}$ decomposes as

$$\overline{\mathfrak{O}} := \mathfrak{O}^{\mathrm{f}} \times \mathfrak{O}^{\infty} := \mathfrak{O} \times \prod_{\sigma \in \Sigma} \sigma^* \mathfrak{O}.$$

Observe that there are properties of $\mathcal{A}$ that can be shown to hold over the analytic part, that does not necessarily hold over the finite part due to the fact that the base field is algebraically closed.

6.2 Divisors

Choose generators $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_r\}$ of $\mathscr{A}$ over $\mathfrak{Z}(\mathscr{A})$, where $r = \mathrm{rk}_{\mathfrak{Z}}(\mathscr{A})$. Let $\mathscr{L}_{\mathfrak{Z}}$ be an invertible subsheaf of $k(\mathfrak{Z}(\mathscr{A}))$ and let $\{D_j\}$ be an affine covering of $\mathfrak{Z}(\mathscr{A})$.

Let $\{\alpha_i\}_{i=1}^r$ be global sections of $\mathscr{L}_{\mathfrak{Z}}$ on Spec $\mathfrak{Z}(\mathscr{A})$. Define elements

$$\boldsymbol{\alpha}_j := \sum_{i=1}^{r} (\alpha_i)_j \boldsymbol{e}_i \in \mathscr{A}(D_j)$$

over each D_j, where we have put $(\alpha_i)_j := \alpha_i(D_j)$. These glue to a global section $\boldsymbol{\alpha}$ over Spec $\mathfrak{Z}(\mathscr{A})$.

Put $\mathscr{L} := \mathscr{A} \cdot \boldsymbol{\alpha} \cdot \mathscr{A}$ which, over D_j, is

$$\mathscr{L}(D_j) = \mathscr{A}(D_j) \cdot \boldsymbol{\alpha}_j \cdot \mathscr{A}(D_j).$$

This defines an invertible A-module over $\mathfrak{Z}(\mathscr{A})$ (in the sense of section 4.6) and hence an element in $\mathrm{Pic}_{\mathfrak{Z}}(\mathscr{A})$. We say that $\mathscr{L} \in \mathrm{Pic}_{\mathfrak{Z}}(\mathscr{A})$ as defined above is a *line sheaf* (resurrecting Serge Lang's terminology which I like) over $\mathscr{A}$. Notice that the restriction of $\mathscr{L}$ to $\mathfrak{Z}(\mathscr{A})$,

$$\mathscr{L}|_{\mathfrak{Z}(\mathscr{A})} = \mathfrak{Z}(\mathscr{A}) \cdot \boldsymbol{\alpha} \cdot \mathfrak{Z}(\mathscr{A}),$$

is not $\mathscr{L}_{\mathfrak{Z}}$. For one thing, $\mathscr{L}|_{\mathfrak{Z}(\mathscr{A})}$ does not have rank one.

Let S be the support of $\mathscr{L}|_{\mathfrak{Z}(\mathscr{A})}$. The *support*, $|\mathscr{L}|$, of $\mathscr{L}$ is the fibre $\Psi^{-1}(S)$.

Definition 6.3 Let the data above be given.

(i) A *Cartier divisor* on $\mathfrak{X}_A$ is a pair $(\mathscr{L}, s_{\mathscr{L}})$, where $\mathscr{L}$ is a line sheaf on $\mathfrak{X}_A$ and $s_{\mathscr{L}}$ a global section of $\mathscr{L}$, as constructed above.
(ii) An *Arakelov–Cartier* divisor on $\overline{\mathfrak{X}}_A$ is a pair $\overline{\mathscr{L}} := \left(\mathscr{L}^{\mathrm{f}}, \mathscr{L}^{\infty}\right)$, where $\mathscr{L}^{\mathrm{f}}$, is a Cartier divisor on the finite part $\mathfrak{X}_A^{\mathrm{f}}$ and $\mathscr{L}^{\infty}$ a Cartier divisor on the analytic part $\mathfrak{X}_A^{\infty}$.
(iii) The divisor $(\overline{\mathscr{L}}, s_{\overline{\mathscr{L}}})$, where $s_{\overline{\mathscr{L}}} := (s_{\mathscr{L}^{\mathrm{f}}}, s_{\mathscr{L}^{\infty}})$, is *effective* if all $\mathscr{L}_{\mathfrak{Z}}$ is effective.

Let $X \xrightarrow{f}$ Spec $\mathfrak{o}$ be an *arithmetic surface* (i.e., X has relative dimension one over $\mathfrak{o}$), then a *vertical (or fibral) divisor* $\mathtt{D}$ is a divisor included in a fibre $X \otimes_{\mathfrak{o}} k(\mathfrak{q})$, for $\mathfrak{q} \in$ Spec $\mathfrak{o}$. Equivalently, $\mathtt{D}$ is vertical if $f(\mathtt{D}) = \{\mathfrak{q}\}$. In addition, a divisor $\mathtt{D} \subset X$ such that $f(\mathtt{D}) =$ Spec $\mathfrak{o}$ is called a *horizontal divisor*. Equivalently, $\mathtt{D}$ is horizontal if it is the Zariski closure of a closed point on the generic fibre of X.

In terms of affine algebras a prime divisor is a codimension-one prime $\mathfrak{p} \subset B$, i.e., a prime such that $\dim(B/\mathfrak{p}) = 1$. Hence, $\mathfrak{p}$ is fibral if $\mathfrak{p} = f^a(\mathfrak{q})$, for some $\mathfrak{q} \in$ Spec $\mathfrak{o}$; $\mathfrak{p}$ is horizontal if $\mathfrak{p} = f^a(\mathfrak{o})$. Here f^a is the to f associated algebraic map.

Recall that $\dim(A)$ denotes the classical Krull dimension, i.e., the supremum of all chains of 2-sided prime ideals in A. Hence, saying that a prime $\mathfrak{p}$ has codimension n means that $\dim(A/\mathfrak{p}) = n$.

Since A is prime, Posner's theorem [18, Theorem 6.5] implies that there is a central simple algebra $\mathrm{Q}(A)$ in which A is a maximal order. In fact,

$$\mathrm{Q}(A) = A \otimes_{\mathfrak{Z}(A)} \mathfrak{Z}(A)_{(0)} = A \otimes_{\mathfrak{Z}(A)} k(\mathfrak{Z}(A)),$$

where $\mathfrak{Z}(A)_{(0)}$ denotes localisation at the generic point, and $k(\mathfrak{Z}(A))$ the field of quotients (and these two are the same). The ring $\mathrm{Q}(A)$ is a right ring of fractions for A.

Let $\mathfrak{p} \subset A$ be a 2-sided codimension-one prime in A, and put $\mathfrak{p}_{\mathfrak{z}} := \mathfrak{p} \cap \mathfrak{Z}(A)$ with residue class field $k(\mathfrak{p}) := k(\mathfrak{p}_{\mathfrak{z}})$. Let ϱ be the representation $\varrho : A \to \mathrm{End}_{k(\mathfrak{p})}(A/\mathfrak{p})$. The versal family of ϱ is

$$\hat{\varrho} : A \longrightarrow \mathrm{End}_{k(\mathfrak{p})}(A/\mathfrak{p}) \otimes_{k(\mathfrak{p})} \hat{H}_{A/\mathfrak{p}},$$

where $\hat{H}_{A/\mathfrak{p}}$ is the pro-representing hull of ϱ. Recall that this is a local (non-commutative) ring. Let $\mathfrak{m}_{\hat{H}}$ be the maximal ideal. We define an order function associated with $\mathfrak{p}$ as

$$\mathrm{ord}_{\mathfrak{p}}(f) := \min \left\{ n \in \mathbb{Z}_{\geq 0} \mid \hat{\rho}|_{\hat{H}}(f) \in \mathfrak{m}^n_{\hat{H}} \right\}, \quad f \in A,$$

extended to $fg^{-1} \in \mathrm{Q}(A)$ as usual by

$$\mathrm{ord}_{\mathfrak{p}}(fg^{-1}) := \mathrm{ord}_{\mathfrak{p}}(f) - \mathrm{ord}_{\mathfrak{p}}(g).$$

Remark 6.1 When A is commutative, the above construction reduces to the classical commutative situation. For instance, when A is commutative, we have an isomorphism

$$\hat{\mathcal{O}}_{\mathrm{Spec}\, A, \mathfrak{p}} \simeq \hat{H}_{A/\mathfrak{p}}$$

and so $\mathrm{ord}_{\mathfrak{p}}$-function reduces to the commutative order function.

Definition 6.4 Let $X = \mathrm{Spec}\, B \to \mathrm{Spec}\, \mathfrak{o}$ be an affine scheme over $\mathfrak{o}$ and let $\mathfrak{X}_A^{(1)}$ denote the set of 2-sided codimension-one primes in A.

(a) A *Weil divisor* on $\mathfrak{X}_A$ is a formal sum

$$\mathtt{D} = \sum_{\mathtt{P} \in \mathfrak{X}_A^{(1)}} n_{\mathtt{P}} \cdot \mathtt{P}, \quad n_{\mathtt{P}} \in \mathbb{Z},$$

where all but finitely many $n_{\mathtt{P}}$ are zero. The divisor is *effective* if all $n_{\mathtt{P}} \geq 0$. The primes $\mathtt{P} \in \mathtt{D}$ such that $n_{\mathtt{P}} \neq 0$ are the *prime divisors* of $\mathtt{D}$. Extending the definition to the analytic part in the obvious way, we can speak of *Arakelov–Weil divisors*.

(b) If $\mathrm{Spec}\, \mathfrak{Z}(A)$ is an arithmetic surface, a Weil divisor $\mathtt{D}$ on $\mathfrak{X}_A$ is *vertical (horizontal)* if the intersection $\mathtt{D} \cap \mathfrak{Z}(A)$ is a vertical (horizontal) Weil divisor on $\mathfrak{Z}(A)$.

(c) Let $(\mathscr{L}, s_{\mathscr{L}})$ be a Cartier divisor on $\mathfrak{X}_A$. Then the associated *Weil divisor* is the formal sum

$$\mathrm{Weil}(\mathscr{L}) := \sum_{\mathtt{P} \in \mathfrak{X}_A^{(1)}} \mathrm{ord}_{\mathtt{P}}(s_{\mathscr{L}})\mathtt{P},$$

where $\mathfrak{X}_A^{(1)}$ denotes the set of (2-sided) codimension-one primes of A.

When viewing $\mathtt{P}$ as a prime ideal we write it as $\mathfrak{p}$, and vice versa.

Let $\boldsymbol{\xi} : A \to S$ be an étale S-rational point with associated map

$$A/\ker(\boldsymbol{\xi} \circ \iota) \xrightarrow{(\alpha_i)} \prod_{i=1}^{n} (A/\mathfrak{p}_i)^{n_i}, \quad \ker(\boldsymbol{\xi} \circ \iota) \subseteq \mathfrak{p}_i, \quad n_i \geq 1.$$

The underlying points are $\bar{\xi}_i = A/\mathfrak{p}_i$. Put

$$C_{\boldsymbol{\xi}} := \left\{ \mathfrak{p}_i \mid \ker(\boldsymbol{\xi} \circ \iota) \subset \mathfrak{p}_i, \ \mathrm{codim}(\mathfrak{p}_i) = 1 \right\}.$$

Then $\boldsymbol{\xi}$ defines a Weil divisor

$$\mathtt{D}_{\boldsymbol{\xi}} = \sum_{\mathfrak{p}_i \in C_{\boldsymbol{\xi}}} n_i \mathtt{P}_i,$$

where $\mathtt{P}_i$ is the divisor corresponding to $\mathfrak{p}_i$. Conversely, given a Weil divisor

$$\mathtt{D} = \sum_{\mathtt{P} \in \mathfrak{X}_A^{(1)}} n_{\mathtt{P}} \mathtt{P},$$

we can define an S-rational point:

$$\boldsymbol{\xi}_{\mathtt{D}} : A \longrightarrow S, \quad \text{with} \quad S = \prod_{i=1}^{s} (A/\mathfrak{p}_i)^{n_i},$$

with $\mathfrak{p}_i$ once again corresponding to $\mathfrak{p}_i$. Hence, Weil divisors are essentially étale rational points where the underlying points are of codimension one.

We can define linear equivalence by restricting to the centre. Put

$$\mathtt{D} := \sum_{\mathtt{P} \in \mathfrak{X}_A^{(1)}} d_{\mathtt{P}} \cdot \mathtt{P}, \quad \text{and} \quad \mathtt{E} := \sum_{\mathtt{P} \in \mathfrak{X}_A^{(1)}} e_{\mathtt{P}} \cdot \mathtt{P}, \qquad d_{\mathtt{P}}, e_{\mathtt{P}} \in \mathbb{Z}.$$

The restriction to the centre gives

$$\mathtt{D}_{\mathfrak{Z}} := \sum_{\mathtt{P} \in \mathfrak{X}_A^{(1)}} d_{\mathtt{P}} \cdot (\mathtt{P} \cap \mathfrak{Z}(A))$$

and similarly with $\mathtt{E}$. If $\mathfrak{p}$ has codimension one so does $\mathfrak{p} \cap \mathfrak{Z}(A)$ (follows from the going up theorem [18, Theorem 8.14(ii)]), so the above defines a Weil divisor on $\mathfrak{Z}(A)$. We say that $\mathtt{D}$ and $\mathtt{E}$ is *linearly equivalent*, writing $\mathtt{D} \sim \mathtt{E}$, if $\mathtt{D}_{\mathfrak{Z}} \sim \mathtt{E}_{\mathfrak{Z}}$.

The free abelian group of all divisors on $\mathfrak{X}_A$, modulo those linearly equivalent to the zero divisor, is the (first) *Chow group*, $\mathrm{CH}^1(\mathfrak{X}_A)$. This extends naturally to the analytic part, allowing us to define $\mathrm{CH}^1(\overline{\mathfrak{X}}_A)$ in the obvious way.

Using the linear equivalence of divisors, we can define the same notion for Cartier divisors. Let $(\mathscr{L}, s_{\mathscr{L}})$ and $(\mathscr{K}, s_{\mathscr{K}})$ be two Cartier divisors on $\mathfrak{X}_A$. We then define $(\mathscr{L}, s_{\mathscr{L}})$ and $(\mathscr{K}, s_{\mathscr{K}})$ to be linearly equivalent if the associated Weil divisors are. In this way we can define a group of Cartier divisors $\mathrm{Cart}(\mathfrak{X}_A)$, using the structure on $\mathrm{CH}^1(\mathfrak{X}_A)$. As above this extends to the analytic part and we can introduce $\mathrm{Cart}(\overline{\mathfrak{X}}_A)$.

6.3 Intersection products

In this section we make a rudimentary attempt at defining an intersection theory on $\mathfrak{X}_A$. A more sophisticated method involving, among other things, the infinite part (i.e., true Arakelov theory) should possibly be discussed at a later stage.

We will write intersection products on $\mathfrak{X}_A$ as $\odot$.

Let $\mathtt{D}$ and $\mathtt{E}$ be prime Weil divisors. If $\mathtt{D}_{\mathfrak{Z}}, \mathtt{E}_{\mathfrak{Z}} \subset \mathtt{azu}(A)$ we define the intersection product of $\mathtt{D}$ and $\mathtt{E}$ in $\mathfrak{X}_A$ as the étale rational point

$$\mathtt{D} \odot \mathtt{E} := \left(\boldsymbol{\xi} : A \longrightarrow \frac{A}{A(\mathtt{D}_{\mathfrak{Z}} \cap \mathtt{E}_{\mathfrak{Z}})A} \right).$$

The intersection number is defined as

$$\mathsf{i}(\mathtt{D}, \mathtt{E}) := \sum_{\mathfrak{s} \in \mathtt{D}_{\mathfrak{Z}} \cap \mathtt{E}_{\mathfrak{Z}}} \text{length}\left(\frac{A \otimes_{\mathfrak{Z}(A)} \mathcal{O}_{\mathfrak{Z}(A),\mathfrak{s}}}{A(\mathtt{D}_{\mathfrak{Z}} \cap \mathtt{E}_{\mathfrak{Z}})A} \right),$$

where

$$\mathsf{i}_{\mathfrak{s}}(\mathtt{D}, \mathtt{E}) := \text{length}\left(\frac{A \otimes_{\mathfrak{Z}(A)} \mathcal{O}_{\mathfrak{Z}(A),\mathfrak{s}}}{A(\mathtt{D}_{\mathfrak{Z}} \cap \mathtt{E}_{\mathfrak{Z}})A} \right)$$

is the local intersection number at $\mathfrak{s}$. This is well-defined since $A \otimes_{\mathfrak{Z}(A)} \mathcal{O}_{\mathfrak{Z}(A),\mathfrak{s}}$ is an Azumaya algebra of finite rank over $\mathcal{O}_{\mathfrak{Z}(A),\mathfrak{s}}$ and the set $\mathtt{D}_{\mathfrak{Z}} \cap \mathtt{E}_{\mathfrak{Z}}$ is finite.

Let's look at the ramification locus and make the following definition.

Definition 6.5 Suppose $\mathtt{D}$ and $\mathtt{E}$ are as above but such that $\mathtt{D}_{\mathfrak{Z}} \cap \mathtt{E}_{\mathfrak{Z}} \subset \mathtt{ram}(A)$. Assume first that the intersection is one point $\mathfrak{p}$ and that

$$\mathsf{N} := \Psi^{-1}(\mathfrak{p}) := \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_s\} \qquad \text{(as ideals)}.$$

Then N defines an étale S-rational point

$$\boldsymbol{\xi} : A \to S, \quad \text{where} \quad S = \prod_{i=1}^{s} A/\mathfrak{m}_i.$$

We now define

$$\mathtt{D} \odot \mathtt{E} := \boldsymbol{\xi}$$

and

$$\mathsf{i}_{\mathfrak{p}}(\mathtt{D}, \mathtt{E}) := \dim_{k(\mathfrak{p})}\big(\hat{\mathcal{O}}_{\mathsf{N}} \big/ \hat{\rho}(\ker \boldsymbol{\xi})\big),$$

where

$$\hat{\rho} : A \to \hat{\mathcal{O}}_{\mathsf{N}}$$

is the versal family of N. Since $\hat{\mathcal{O}}_{\mathsf{N}}$ is semi-local the algebra $\hat{\mathcal{O}}_{\mathsf{N}} \big/ \hat{\rho}(\ker \boldsymbol{\xi})$ is finite-dimensional over $k(\mathfrak{p})$, this definition is well-defined.

The total intersection number is defined in the obvious way:

$$\mathsf{i}(\mathtt{D}, \mathtt{E}) := \sum_{\mathfrak{s} \in \mathtt{D}_{\mathfrak{Z}} \cap \mathtt{E}_{\mathfrak{Z}}} \mathsf{i}_{\mathfrak{p}}(\mathtt{D}, \mathtt{E}).$$

The definition extends naturally to the case in which $\mathtt{D}_{\mathfrak{Z}} \cap \mathtt{E}_{\mathfrak{Z}} \subset \mathtt{ram}(A)$ is more than one point.

Note that we assume here that the intersection happens on a finite fibre, so that the ring $\hat{\mathcal{O}}$ is defined. The same definition extends to the generic and analytic fibre in the natural manner.

Remark 6.2 Clearly, since the above definitions involve deformation theory, intersections are quite difficult to compute as defined above. However, I feel that the above is the "correct" one in the present context. Also it should be said that there are other, more general and abstract, versions of intersection theory on non-commutative spaces (for instance [11] in the case of non-commutative surfaces). However, the definition of non-commutative spaces in those versions are global, whereas the approach taken in this paper is fundamentally local. It seems to me that the global approach is not particularly suited for applications i arithmetic.

7 Examples of non-commutative arithmetic spaces

7.1 Non-commutative quotient spaces

Let X be a quasi-projective scheme and G a *finite* group acting on X. Since X is quasi-projective and G finite, the quotient X/G exists as a quasi-projective scheme.

The group G acts on the structure sheaf $\mathcal{O}_X$ such that $G \cdot \mathcal{O}_X(U) \subset \mathcal{O}_X(U)$ and so we can look at the $\mathcal{O}_X$-algebra $\mathcal{A} := \mathcal{O}_X\langle G \rangle$. This is the finite $\mathcal{O}_X$-algebra defined as

$$\mathcal{A} := \bigoplus_{\tau \in G} \mathcal{O}_X \cdot \tau, \quad \tau y = \tau(y)\tau, \quad \text{for all } y \in \mathcal{O}_X.$$

The centre of $\mathcal{A}$ is

$$\mathfrak{Z}(\mathcal{A}) = \mathcal{O}_X^G$$

and $\mathcal{A}$ is finite as a module over $\mathfrak{Z}(\mathcal{A})$ and hence a PI-algebra over X/G. Observe that it is not a PI-algebra over X since $\mathcal{O}_X^G \subset \mathcal{O}_X$.

It is a general fact that the simple $\mathcal{A}$-modules are in one-to-one correspondence with the set of orbits of X under G, in other words, the closed points on X/G. Hence, in this way $\mathfrak{X}_{\mathcal{A}}$ can be viewed as "non-commutative thickening" of X/G.

Also, even if X is not quasi-projective, the quotient X/G exists as a *Deligne–Mumford stack*. If G is not finite the quotient exists as an *Artin* (or *algebraic*) *stack*. Normally, stack quotients are denoted $[X/G]$.

In view of this, we denote the non-commutative space associated with $\mathcal{A}$ as

$$[\![X/G]\!] := \mathfrak{X}_{\mathcal{A}}$$

and call this the *non-commutative quotient* of X modulo G and X/G the *coarse space*. Observe that this is commutative.

Remark 7.1 Let A be a noetherian prime ring which is finite over its centre. Then A is *homologically homogeneous* (*hom-hom*, for short) if A has finite global dimension and, for every pair $\mathfrak{M}_1, \mathfrak{M}_2 \in \operatorname{Max} A$ such that $\mathfrak{M}_1 \cap \mathfrak{Z}(A) = \mathfrak{M}_2 \cap \mathfrak{Z}(A)$, the simple modules $A/\mathfrak{M}_1$ and $A/\mathfrak{M}_2$ have the same projective dimension. Observe that this is a natural extension of regularity in the commutative sense. It is known (see [23, Thm. 5.6]) that hom-hom implies Auslander-regularity (which we won't define) and in the graded case, Artin–Schelter regularity (which we won't define either). If A includes a field it is also Cohen–Macaulay.

Now, a *non-commutative crepant resolution* of a commutative ring R is any ring Δ such that $\Delta \simeq \operatorname{End}_R(M)$ where M is a reflexive R-module. Let V be a finite rank free $\mathfrak{o}$-module with a linear action of G. Then

$$R\langle G \rangle \simeq \operatorname{End}_{R^G}(\operatorname{Sym}(V))$$

is a non-commutative crepant resolution of R^G. Therefore, $[\![X/G]\!]$ can be viewed as a non-commutative desingularisation of R^G.

We will now look a couple of quotient spaces and an example with an order over an arithmetic surface.

7.2 The plane $\mathbb{Z}/3$-quotient singularity

Recall that a point on a non-commutative space $\mathfrak{X}_A$ is a representation $\varrho : A \to \operatorname{End}(M)$ such that $\ker \varrho$ is a prime ideal. Remember that the point is a closed étale point if there are maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_s$ such that $\ker(\boldsymbol{\varrho} \circ \iota) \subseteq \mathfrak{m}_i$. This implies in particular, that $\ker(\boldsymbol{\varrho} \circ \iota)$ is maximal. The underlying points are the simple algebras $\bar{\varrho}_i = A/\mathfrak{m}_i$.

Let ζ be a primitive third root of unity and assume that $\mathbb{Z}[\zeta] \subseteq \mathfrak{o}$. We let $\mu_3 = \langle \sigma \rangle$ act on $\bar{\mathfrak{o}}[x, y]$ as

$$\sigma: \quad x \mapsto \zeta x, \quad y \mapsto \zeta^2 y.$$

Put

$$\widehat{A} := \bar{\mathfrak{o}}[x, y]\langle \mu_3 \rangle = \frac{\bar{\mathfrak{o}}[x, y]\langle \sigma \rangle}{(\sigma x - \zeta x \sigma,\ \sigma y - \zeta^2 y \sigma,\ \sigma^3 = 1)}.$$

The centre $\mathfrak{Z}(\widehat{A})$ is

$$\widehat{A}^{\mu_3} = \bar{\mathfrak{o}}[x^3, xy, y^3] = \bar{\mathfrak{o}}[r, s, t]/(t^3 - rs), \quad r := x^3,\ s := y^3,\ t := xy,$$

and has a singularity at the origin across all fibres.

Let $\boldsymbol{\varrho} : \widehat{A} \to \operatorname{End}_{\bar{\mathfrak{o}}}(M)$ be a point on $\overline{\mathfrak{X}}_A$. The point restricts to a point, $\boldsymbol{\varrho}_{k(\mathfrak{p})}$, on each fibre for every $\mathfrak{p} \in \operatorname{Spec} \widehat{A}$. Note, however, that $\widehat{A}_{k(\mathfrak{p})}$ degenerates to $k(\mathfrak{p})[x, y, \sigma]$ if there are no non-trivial third roots of unity in $k(\mathfrak{p})$.

For simplicity of notation, let's fix a prime $\mathfrak{p} \in \operatorname{Spec} \bar{\mathfrak{o}}$ such that $k := k(\mathfrak{p})$ includes a non-trivial third root of unity. Accordingly, we write A instead $\widehat{A}$. Note that $\mathfrak{p}$ need not be a finite prime.

Now, let $\boldsymbol{p} := (x - a, y - b)$ be a k-point on $\operatorname{Spec} k[x, y]$. The orbit of $\boldsymbol{p}$ under μ_3 is the k-scheme

$$\operatorname{Spec}\left(\frac{k[x, y]}{(x - a, y - b)} \times \frac{k[x, y]}{(x - \zeta a, y - \zeta^2 b)} \times \frac{k[x, y]}{(x - \zeta^2 a, y - \zeta b)} \right).$$

This corresponds to the A-module

$$\varrho : A \to \operatorname{End}_k(M_{(a,b)}), \quad M_{(a,b)} := k\boldsymbol{e}_1 \oplus k\boldsymbol{e}_2 \oplus k\boldsymbol{e}_3,$$

with actions

$$\varrho(x) = \begin{pmatrix} a & 0 & 0 \\ 0 & \zeta^{-1} a & 0 \\ 0 & 0 & \zeta^{-2} a \end{pmatrix}, \quad \varrho(y) = \begin{pmatrix} b & 0 & 0 \\ 0 & \zeta^{-2} b & 0 \\ 0 & 0 & \zeta^{-1} b \end{pmatrix}, \quad \varrho(\sigma) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The kernel of ϱ is generated by $\mathfrak{m} := (x^3 - a, y^3 - b, \sigma^3 - 1)$. If $ab \neq 0$ the algebra $A/\mathfrak{m}$ is a central simple algebra and so ϱ defines a closed point on $\mathfrak{X}_A$. On the other hand, if $a = 0$ (or $b = 0$) $A/\mathfrak{m}$ is not simple and so ϱ is an open point. The underlying closed points are $(x^3 - a, y, \sigma^3 - 1)$ and $(x, y^3 - b, \sigma^3 - 1)$.

Similary, we take $N_{(u,v)} := k\boldsymbol{e}'_1 \oplus k\boldsymbol{e}'_2 \oplus k\boldsymbol{e}'_3$ with

$$\varrho'(x) = \begin{pmatrix} u & 0 & 0 \\ 0 & \zeta^{-1} u & 0 \\ 0 & 0 & \zeta^2 u \end{pmatrix}, \quad \varrho'(y) = \begin{pmatrix} v & 0 & 0 \\ 0 & \zeta^{-2} v & 0 \\ 0 & 0 & \zeta^{-1} v \end{pmatrix},$$

and $\varrho'(\sigma) = \varrho(\sigma)$.

Put

$$\delta(x) := \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \qquad \delta(y) := \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}$$

and

$$\delta(\sigma) := \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}$$

From the relation $\delta(\sigma^3 - 1) = 0$ follows

$$\begin{aligned} s_{11} &= -s_{22} - s_{33} \\ s_{12} &= -s_{23} - s_{31} \\ s_{13} &= -s_{21} - s_{32} \end{aligned} \tag{8}$$

Since x and y commutes we find that $\delta(xy - yx) = 0$ leads to (after some simplifications)

$$\begin{aligned} (b - v)x_{11} &= (a - u)y_{11} \\ (\zeta b - v)x_{12} &= (\zeta^{-1}a - u)y_{12} \\ (\zeta^{-1}b - v)x_{13} &= (\zeta a - u)y_{13} \\ (b - \zeta v)x_{21} &= (a - \zeta^{-1}u)y_{21} \\ \zeta^2(b - v)x_{22} &= (a - u)y_{22} \\ (b - \zeta^2 v)x_{23} &= (\zeta^2 a - u)y_{23} \\ (b - \zeta^{-1}v)x_{31} &= (a - \zeta u)y_{31} \\ (\zeta^2 b - v)x_{32} &= (a - \zeta^2 u)y_{32} \\ (b - v)x_{33} &= \zeta^2(a - u)y_{33} \end{aligned} \tag{9}$$

Similarly, $\delta(\sigma x - \zeta x\sigma) = 0$ gives, (again after simplifications)

$$\begin{aligned} x_{11} &= \zeta x_{22} - (a - u)s_{21} \\ x_{31} &= \zeta x_{12} - (a - \zeta u)(s_{22} + s_{33}) \\ x_{32} &= \zeta x_{13} - \zeta(\zeta a - u)(s_{23} + s_{31}) \\ x_{33} &= \zeta x_{22} - (a - u)(s_{21} + s_{32}) \\ x_{21} &= \zeta^2 x_{13} - (a - \zeta^2 u)s_{23} \\ x_{23} &= \zeta^2 x_{12} - \zeta(a - \zeta u)s_{22} \end{aligned}$$

and, by symmetry, $\delta(\sigma y - \zeta^2 y\sigma) = 0$,

$$\begin{aligned} y_{11} &= \zeta y_{22} - (b - v)s_{21} \\ y_{31} &= \zeta y_{12} - (b - \zeta v)(s_{22} + s_{33}) \\ y_{32} &= \zeta y_{13} - \zeta(\zeta b - v)(s_{23} + s_{31}) \\ y_{33} &= \zeta y_{22} - (b - v)(s_{21} + s_{32}) \\ y_{21} &= \zeta^2 y_{13} - (b - \zeta^2 v)s_{23} \\ y_{23} &= \zeta^2 y_{12} - \zeta(b - \zeta v)s_{22}. \end{aligned} \tag{10}$$

We find that we can chose x_{12}, x_{13} and x_{22} as parameters from $\delta(x)$ and s_{21}, s_{22}, s_{23}, s_{31}, s_{32} and s_{33} from $\delta(\sigma)$. For a generic point $(u, v) = (a, b)$ (which is (a^3, b^3, ab) in Spec $\mathfrak{Z}(A)$), we can additionally choose x_{11} and y_{11} as parameters.

Since σ only scrambles the entries in a matrix upon multiplication (from the left and right), we easily see that the dimension of the inner derivations is 9. Therefore, for a generic point $(u, v) = (a, b)$ the dimension is two as it should be.

However, for specific choices of points, the ext-dimensions are higher. These are the open points mentioned above. In fact, the images of the coordinate axes from $\mathbb{A}^2$ to Spec $\mathfrak{Z}(A)$ is $\mathtt{ram}(A)$:

$$\mathtt{ram}(A) = \mathrm{Spec}\,(k[r, s, t]/(t^3 - rs, s, t)) \cup \mathrm{Spec}\,(k[r, s, t]/(t^3 - rs, r, t)).$$

Indeed, put $b = v = 0$ and, say, $u = \zeta a$. Then $u^3 = a^3$ so both $(a, 0)$ and $(\zeta a, 0)$ lie over the same point in $\mathfrak{Z}(A)$. We see that the left-hand side of (9) is zero. Also, in row 3, we find $\zeta(a - a)y_{13} = 0$. This implies that y_{13} is a free parameter. Similarly, it looks like y_{21} and y_{32} would also become free, but from (10) both of these are expressible in terms of y_{13}. Hence we gain one free parameter and so

$$\mathrm{Ext}^1_A\left(M_{(a,0)}, N_{(\zeta a,0)}\right) = k.$$

We easily see that there are 1-dimensional Ext's between all three points above $(a^3, 0)$. By symmetry we find that the same holds for the "y-axis" $(0, b^3)$.

It is quite easy to convince oneself that all deformations are unobstructed.

Put $N_i := M_{(\zeta^{i-1}a,0)}$, $i = 1, 2, 3$. For $\boldsymbol{p}_3 \in \mathtt{ram}(A)$, we can view the fibre $\Psi^{-1}(\boldsymbol{p}_3) = \{N_1, N_2, N_3\}$ as an étale point. Indeed, let $\boldsymbol{p}_3$ correspond to the maximal ideal $\mathfrak{m}$. Then the extension of $\mathfrak{m}_3$ to A splits into three maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$, corresponding to N_1, N_2, N_3, and so $\varrho : A \to \mathrm{End}_k(A/\mathfrak{m}_3)$ is an étale point with $\ker(\varrho \circ \iota) \subseteq \mathfrak{m}_i$. The underlying closed points are then the simple algebras $A/\mathfrak{m}_i$.

We summarise the discussion with the following theorem.

Theorem 7.1 *The non-commutative space* $\left[\!\left[\mathbb{A}^2_{\mathfrak{o}}/\mu_3\right]\!\right]$ *is*

$$\left[\!\left[\mathbb{A}^2_{\mathfrak{o}}/\mu_3\right]\!\right] = (\mathsf{Mod} A, \mathbb{O}),$$

where, for $\boldsymbol{p} \in \mathtt{azu}(A)$*, viewed as both a module and point,*

$$\hat{\mathbb{O}}_{\boldsymbol{p}} = \hat{\mathbb{O}}_{\mathbb{A}^2_{\mathfrak{o}}/\mu_3, \boldsymbol{p}} \simeq \mathrm{End}_k(\boldsymbol{p}) \otimes_k \hat{H}_{\boldsymbol{p}}.$$

Over $\mathtt{ram}(A)$*, we have the étale point* $\mathsf{N} := \{N_1, N_2, N_3\}$ *and so*

$$\hat{H}_{\mathsf{N}} = \begin{pmatrix} k\langle\langle t^1_{11}, t^2_{11}\rangle\rangle & \langle t_{12}\rangle & \langle t_{13}\rangle \\ \langle t_{21}\rangle & k\langle\langle t^1_{22}, t^2_{22}\rangle\rangle & \langle t_{23}\rangle \\ \langle t_{31}\rangle v\langle t_{33}\rangle & k\langle\langle t^1_{33}, t^2_{33}\rangle\rangle & \end{pmatrix},$$

and, with hopefully clear notation,

$$\hat{\mathbb{O}}_{\mathsf{N}} = \mathrm{Hom}_k(\mathsf{N}) \otimes_k \hat{H}_{\mathsf{N}}.$$

The closed points of $\left[\!\left[\mathbb{A}^2_{\mathfrak{o}}/\mu_3\right]\!\right]$ *are stratified as*

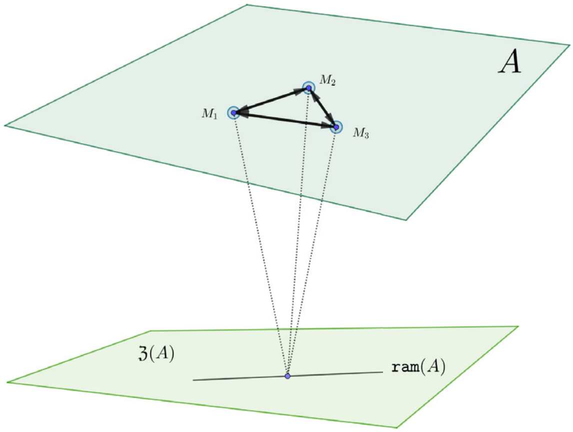
$$\left[\!\left[\mathbb{A}^2_{\mathfrak{o}}/\mu_3\right]\!\right]_n = \mathfrak{X}_{A,n} = (\mathsf{Mod}_n A, \mathbb{O}_n), \quad \textit{where } n = 1, 3,$$

and

$$\left[\!\left[\mathbb{A}^2_{\mathfrak{o}}/\mu_3\right]\!\right]_3 = \mathtt{azu}(A), \quad \textit{and} \quad \left[\!\left[\mathbb{A}^2_{\mathfrak{o}}/\mu_3\right]\!\right]_1 = \mathtt{ram}(A).$$

It is important to observe that the statements made in the theorem are made fibre-by-fibre (anything else is meaningless since everything is trivial across different characteristics). We have chosen not to make this explicit with an awkward notation such as $\left[\!\left[\mathbb{A}^2_{\mathfrak{o}}/\mu_3\right]\!\right] \otimes k(\mathfrak{p})$ or something similar.

Fig. 1 Ramification situation for $[\![\mathbb{A}^2_0/\mu_3]\!]$ on a fibre



7.2.1 Diophantine geometry of $[\![\mathbb{A}^2_0/\mu_3]\!]$

Recall that an étale rational point on $\mathfrak{X}_A$ is an algebra morphism $\boldsymbol{\xi} : A \to S$ such that there is a map

$$A/\ker(\boldsymbol{\xi} \circ \iota) \xrightarrow{(\alpha_i)} \prod_i^s A/\mathfrak{p}_i.$$

For reasons explained at the beginning of Sect. 5.3, it is reasonable to assume that S is semi-simple, and so the $\mathfrak{p}_i$ are all maximal, $\bar{\xi}_i := A/\mathfrak{m}_i$ are simple algebras and (α_i) is an isomorphism.

Let $\mathtt{D}_3$, $\mathtt{E}_3$ and $\mathtt{F}_3$ be the central divisors

$$\mathtt{D}_3 := \{r = a^3, s = t = 0\}, \quad \mathtt{E}_3 = \{r = t = a^3, s = 1\},$$

and

$$\mathtt{F}_3 := \{r = 1, s = t = b^3\}.$$

Observe that $\mathtt{D}_3 \subset \mathtt{ram}(A)$ but $\mathtt{E}_3, \mathtt{F}_3 \subset \mathtt{azu}(A)$.

The intersection between $\mathtt{E}_3$ and $\mathtt{F}_3$ is $\mathtt{E}_3 \cap \mathtt{F}_3 = \{r = s = t = 1\}$ corresponding to the ideal

$$\mathfrak{m} := (r - 1, s - 1, t - 1) \subset \mathfrak{Z}(A).$$

This gives the rational point

$$\mathtt{E} \odot \mathtt{F} = \left(\xi : A \to A/\mathfrak{m} = \frac{A}{(x^3 - 1, y^3 - 1)}\right)$$

which is a cental simple algebra (as it should since the intersection is in $\mathtt{azu}(A)$).

The intersection $\mathtt{D}_3 \cap \mathtt{F}_3$ is inside $\mathtt{ram}(A)$ and corresponds to the ideal

$$\mathfrak{n} := (r - 1, s, t) = (x^3 - 1, y^3, xy).$$

This defines the étale rational point

$$\mathtt{D} \odot \mathtt{F} = \left(\boldsymbol{\xi} : A \longrightarrow \prod_{j=0}^{2} \frac{A}{A(x - \zeta^j, y)A}\right).$$

Observe that

$$\sigma(x-\zeta^j)\zeta^i x\sigma \in A(x-\zeta^j, y)A$$

for all i. This implies (taking $i=-j$) that $\sigma x - x\sigma = 0$ in $A/A(x-\zeta^j, y)A$, implying that $A/A(x-\zeta^j, y)A$ is actually commutative. In fact, as is easily seen, $A/A(x-\zeta^j, y)A = k$. Therefore,

$$\mathtt{D}\odot\mathtt{F} = \Big(\boldsymbol{\xi} : A \longrightarrow k\times k\times k\Big).$$

We leave for the reader to compute the intersection numbers (which is not quite so easy).

The module $M_{(\alpha,\beta)}$, with $\alpha,\beta\in k'$, defines a k'-rational point on $[\![\mathbb{A}^2_{\mathfrak{o}}/\mu_3]\!]$ via the structure morphism

$$\xi : A \to \mathrm{End}_{k'}(M_{(\alpha,\beta)}).$$

In addition, let

$$\xi_3 : k[r,s,t]/(t^3-rs) \longrightarrow k'$$

be a k'-rational point on Spec $\mathfrak{Z}(A)$. Then

$$\boldsymbol{\xi} : A \longrightarrow A/\ker\rho_3$$

is an étale $(A/\ker\rho_3)$-rational point on $[\![\mathbb{A}^2_{\mathfrak{o}}/\mu_3]\!]$. Note that $A/\ker\rho_3$ is a k'-algebra.

Example 7.1 Let ξ_3 be the point corresponding to the ideal

$$\ker\xi_3 := (r-\zeta, s-1, t-\tau),\quad \tau^3=\zeta.$$

Then $k' = k(\sqrt[3]{\zeta})$. Note that $\ker\xi_3\in\mathtt{azu}(A)(k')$ and the lift of ξ_3 to A is the rational point

$$\xi : A \to A/\ker\xi_3 = \frac{A}{\big(x^3-\zeta, y^3-1\big)}.$$

This is a k'-central simple algebra.

Suppose now that ξ_3 is the point corresponding to the ideal

$$\ker\xi_3 := (r-\zeta, s, t)\in\mathtt{ram}(A)(k').$$

Then the lift of ξ_3 to A is

$$\boldsymbol{\xi} : A \to A/\ker\xi_3 = \frac{A}{\big(x^3-\zeta, y\big)}.$$

This defines an étale rational point with underlying points $\bar{\xi}_i$, the points corresponding to the ideals over $\ker\xi_3$ in A. Note that the residue ring of $\boldsymbol{\xi}$ is the étale algebra $E_{\boldsymbol{\xi}} = k'\times k'\times k'$.

Let us finally compute the non-commutative height of a point in $\mathtt{ram}(A)$. The adjacency matrix is

$$E = \begin{pmatrix} 2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}$$

whose eigenvalues are $\{4,1,1\}$. Therefore $H^{\mathtt{nc}}_{k'}(\boldsymbol{\xi}) = 4$.

Observe that $H^{\mathtt{nc}}_{k'}(\boldsymbol{\xi})$ is only dependent on the local data $\hat{H}_{\mathsf{N}}$. The other coordinates in the height vector

$$H^{\mathtt{tot}}_{k',\alpha}(\boldsymbol{\xi}) = \Big(H^3_\alpha(\boldsymbol{\xi}), H^{\mathtt{rep}}_{k'}(\boldsymbol{\xi}), H^{\mathtt{nc}}_{k'}(\boldsymbol{\xi})\Big)$$

are the entities directly related to the coordinates of the point $\boldsymbol{\xi}$. Unfortunately I don't know how to compute this even in the simplest (non-trivial) case.

7.3 A non-commutative thickening of Spec $\mathbb{Z}$

Let $d \equiv 2, 3 \bmod 4$ and square-free. Put $F := \mathbb{Q}[\sqrt{d}]$ and $\mathfrak{o}_F$ its ring of integers. The stated hypothesis on d ensures that $\mathfrak{o}_F = \mathbb{Z}[\sqrt{d}]$, with discriminant $\delta_F = 4d$. We use the presentation $\mathfrak{o}_F = \mathbb{Z}[x]/(x^2 - d)$.

This gives a $\mathbb{Z}/2$-cover $\phi : \operatorname{Spec} \mathfrak{o}_F \to \operatorname{Spec} \mathbb{Z}$, ramified over 2 and the prime divisors of d. Since $|\mathbb{Z}/2| = 2$, we see that the cover is wildly ramified over 2 and tamely ramified for all other ramification points.

Put $\mathfrak{H} := \mathbb{Z}/2$. The orbits of $\mathfrak{H}$ on $\mathfrak{o}_F$ come in three types:

(i) The number of points in the orbit is two. This is the (completely) split case.
(ii) The number of points in the orbit is one, without multiplicity. This is the inert case.
(iii) The number of points in the orbit is one, with multiplicity. This is then the ramified case.

Observe that "point" means *closed* point and that $\operatorname{Spec} \mathfrak{o}_F \big/ \mathfrak{H} = \operatorname{Spec} \mathbb{Z}$.

Accordingly, the corresponding $\mathfrak{o}_F\langle\Gamma\rangle$-modules look very different. Let τ denote the non-trivial element of $\mathfrak{H}$, thus acting as $\tau(a + b\sqrt{d}) = a - b\sqrt{d}$. We look at the different cases in turn. Let $\mathfrak{p} \in \operatorname{Spec} \mathfrak{o}_F$ be a prime over $p \in \operatorname{Spec} \mathbb{Z}$ and put

$$A := \mathfrak{o}_F\langle\mathfrak{H}\rangle \simeq \frac{\mathbb{Z}\langle x, \tau\rangle}{\left(x^2 - d, \tau x + x\tau, \tau^2 - 1\right)}.$$

7.3.1 The split case

Assume that p is split. Hence $(p) = \mathfrak{p}_+\mathfrak{p}_-$ and the orbit of $\mathfrak{p}_+$ (or $\mathfrak{p}_-$, of course) is $\{\mathfrak{p}_+, \mathfrak{p}_-\}$. Since the orbits are precisely the fibres of the covering morphism ϕ, the orbit is completely determined by the underlying prime p and we parametrize the orbits using the quotient $\operatorname{Spec} \mathbb{Z}$.

The A-module corresponding to the orbit is

$$M = \frac{\mathbb{F}_p[x]}{(x - a)}\boldsymbol{e}_1 \oplus \frac{\mathbb{F}_p[x]}{(x + a)}\boldsymbol{e}_2,$$

where $x^2 - d = (x - a)(x + a)$ modulo p. In other words,

$$M = \mathbb{F}_p\boldsymbol{e}_1 \oplus \mathbb{F}_p\boldsymbol{e}_2, \quad \text{with} \quad x \mapsto \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This defines a simple A-module. In particular, it defines a closed point on $\mathfrak{X}_A$.

Any $\delta \in \operatorname{Der}_{\mathbb{Z}}\big(A, \operatorname{End}_{\mathbb{F}_p}(M)\big)$ must satisfy

$$\delta(x^2 - d) = \delta(x^2) = 0, \quad \delta(\tau x + x\tau) = 0 \quad \text{and} \quad \delta(\tau^2 - 1) = \delta(\tau^2) = 0.$$

The first relation leads to

$$\delta(x^2) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 2ax_{11} & 0 \\ 0 & -2ax_{22} \end{pmatrix}$$

implying that $x_{11} = x_{22} = 0$ (since $p \neq 2$). The second equation gives

$$\delta(\tau x + x\tau) = \begin{pmatrix} 2ad_{11} + x_{21} + x_{12} & 0 \\ 0 & -2ad_{22} + x_{21} + x_{12} \end{pmatrix} = \mathbf{0},$$

where we have used that $x_{11} = x_{22} = 0$. Similarly,

$$\delta(\tau^2) = \begin{pmatrix} d_{12} + d_{21} & d_{11} + d_{22} \\ d_{11} + d_{22} & d_{12} + d_{21} \end{pmatrix} = \mathbf{0},$$

implying that $d_{22} = -d_{11}$ and $d_{21} = -d_{12}$. We see that d_{11} can be expressed in terms of x_{12} and x_{21} so d_{11} and d_{22} are determined when x_{12} and x_{21} are fixed. Therefore, we can choose d_{12}, x_{12} and x_{21} as free parameters. A small computation shows that all derivations are inner, i.e., $\dim_{\mathbb{F}_p}(\mathrm{Ad}) = 3$, and so $\mathrm{Ext}^1_A(M, M) = 0$ in the completely split case. Observe that this is relative to $\mathbb{Z}$. Therefore, $\hat{H} = k$ and so

$$\hat{\mathcal{O}}_{\{M\}} = \hat{H} \otimes_k \mathrm{End}_k(M) = \mathrm{End}_k(M).$$

This means that M, as an A-module, is rigid in the deformation-theoretic sense which is certainly reasonable (one cannot "deform" prime numbers). This should certainly apply to the inert case also. Let's show explicitly that this is true.

7.3.2 The inert case

When p is inert, we have $(p) = \mathfrak{p} \in \operatorname{Spec} \mathfrak{o}_F$. This means that $x^2 - d$ is irreducible modulo p. However, even though there is only one point in the orbit, the corresponding module is 2-dimensional. The orbit is the fibre of ϕ so the corresponding module is

$$M = \mathfrak{o}_F \otimes_{\mathbb{Z}} k(p) = \mathfrak{o}_F/(p) = \frac{\mathbb{Z}[x]/(x^2 - d)}{(p)} = \mathbb{F}_p[x]/(x^2 - d) = \mathbb{F}_p \boldsymbol{e}_1 \oplus \mathbb{F}_p \boldsymbol{e}_2.$$

The action of x is given as

$$x \cdot \boldsymbol{e}_1 = \boldsymbol{e}_2, \quad \text{and} \quad x \cdot \boldsymbol{e}_2 = d\boldsymbol{e}_1.$$

The action of τ is slightly trickier. Observe first that it cannot be the identity since p is not ramified. However, M is a quadratic extension of $\mathbb{F}_p$ and τ reduces to the non-trivial extension of this extension so must be given by $\tau(\boldsymbol{e}_1) = \boldsymbol{e}_1$ and $\tau(\boldsymbol{e}_2) = -\boldsymbol{e}_2$ modulo p. On matrix form we thus have

$$x \mapsto \begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \tau \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is a simple A-module, i.e., a closed point. Notice that the matrix corresponding to x does not have any eigenvectors over $\mathbb{F}_p$ (since $x^2 - d$ is irreducible) so x cannot preserve any 1-dimensional subspace of M.

The same type of calculation as in the previous case shows that M is rigid here also, i.e., $\mathrm{Ext}^1_A(M, M) = 0$ (implying that $\hat{H} = k$), hence

$$\hat{\mathcal{O}}_{\{M\}} = \mathrm{End}_k(M)$$

in the inert case also.

7.3.3 The ramified case

Suppose now that $p \mid d$. Then

$$N_3 := \mathfrak{o}_F/(p) = \frac{\mathbb{Z}[x]/(x^2 - d)}{(p)} = \mathbb{F}_p[x]/(x^2) = \mathbb{F}_p \boldsymbol{e}_1 \oplus \mathbb{F}_p \boldsymbol{e}_2,$$

with $x \cdot \boldsymbol{e}_1 = \boldsymbol{e}_2$ and $x \cdot \boldsymbol{e}_2 = 0$. The induced action of τ on N_3 is $\tau(\boldsymbol{e}_1) = \boldsymbol{e}_1$, $\tau(\boldsymbol{e}_2) = \boldsymbol{e}_2$. This is an indecomposable module, but not simple. In other words, it does *not* define a closed point.

The composition series $\mathbb{F}_p\boldsymbol{e}_2 \subset \mathbb{F}_p\boldsymbol{e}_1 \oplus \mathbb{F}_p\boldsymbol{e}_2$ gives the two simple modules

$$N_1 := \mathbb{F}_p\boldsymbol{e}_2, \quad N_2 := (\mathbb{F}_p\boldsymbol{e}_1 \oplus \mathbb{F}_p\boldsymbol{e}_2)/\mathbb{F}_p\boldsymbol{e}_2 \simeq \mathbb{F}_p\boldsymbol{e}$$

where $x\boldsymbol{e} = 0$. Observe that $N_1 \simeq N_2$. This should be interpreted as N_3 including two isomorphic, but distinct, closed points that can't be separated.

Let $\delta : A \to \mathrm{Hom}_{\mathbb{F}_p}(N_1, N_2) = \mathrm{End}_{\mathbb{F}_p}(N_1)$ be a derivation. We find

$$\delta(x^2)\boldsymbol{e} = \delta(x)x\boldsymbol{e} + x\delta(x)\boldsymbol{e} = 0, \qquad \delta(\tau^2)\boldsymbol{e} = \delta(\tau)\tau\boldsymbol{e} + \tau\delta(\tau)\boldsymbol{e} = 2d_\tau\boldsymbol{e},$$

the first one following since $x\boldsymbol{e} = 0$. Take $\theta \in \mathrm{End}_{\mathbb{F}_p}(N_1)$. Then,

$$(\theta x - x\theta)\boldsymbol{e} = 0, \quad \text{and} \quad (\theta\tau - \tau\theta)\boldsymbol{e} = 0,$$

so $\dim(\mathrm{Ad}) = 0$. Consequently,

$$\mathrm{Ext}^1_A(N_1, N_2) = \mathrm{Ext}^1_A(N_1, N_1) = \begin{cases} k, & \text{if } p \neq 2, \text{ and} \\ k^2, & \text{if } p = 2. \end{cases}$$

Now, let $\delta : A \to \mathrm{Hom}_{\mathbb{F}_p}(N_2, N_3)$. Put $\delta(x)\boldsymbol{e} = d_1\boldsymbol{e}_1 + d_2\boldsymbol{e}_2$ and $\delta(\tau)\boldsymbol{e} = t_1\boldsymbol{e}_1 + t_2\boldsymbol{e}_2$. Then we find

$$\delta(x^2)\boldsymbol{e} = d_1\boldsymbol{e}_2, \qquad \delta(\tau^2)\boldsymbol{e} = 2t_1\boldsymbol{e}_1 + 2t_2\boldsymbol{e}_2.$$

We can thus choose d_2 and t_2 as free parameters if $p \neq 2$ and, additionally, t_2 as free when $p = 2$. Computing the inner derivations we find that these are 1-dimensional so

$$\mathrm{Ext}^1_A(N_2, N_3) = \begin{cases} k, & \text{if } p \neq 2, \text{ and} \\ k^2, & \text{if } p = 2. \end{cases}$$

In the other direction, we can compute

$$\mathrm{Ext}^1_A(N_3, N_2) = \begin{cases} 0, & \text{if } p \neq 2, \text{ and} \\ k^2, & \text{if } p = 2. \end{cases}$$

Finally, we compute $\mathrm{Ext}^1_A(N_3, N_3)$.

We have

$$\varrho(x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \varrho(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $\delta \in \mathrm{Der}_{\mathbb{F}_p}(A, \mathrm{End}_{\mathbb{F}_p}(N_3))$, put

$$\delta(x) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad \text{and} \quad \delta(\tau) = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}.$$

From $\delta(x^2) = 0$, we find that $x_{12} = 0$, $x_{22} = -x_{11}$ and x_{21} free; from $\delta(\tau^2) = 0$ we get

$$2s_{11} = 2s_{12} = 2s_{21} = 2s_{22} = 0.$$

Assume first, that $p \neq 2$.

Then $s_{11} = s_{12} = s_{21} = s_{22} = 0$ and a general derivation can be written as

$$\delta = \begin{pmatrix} d_{11} & 0 \\ d_{21} & -d_{11} \end{pmatrix}.$$

Therefore, an inner derivation comes from an element $\theta \in \mathrm{End}_{\mathbb{F}_p}(N_3)$ on the form

$$\theta = \begin{pmatrix} \theta_{11} & 0 \\ \theta_{21} & -\theta_{11} \end{pmatrix}.$$

Computing $\theta x - x\theta$ we find that $2\theta_{11} = 0$, and so $\dim(\mathrm{Ad}) = 1$. Consequently, $\mathrm{Ext}^1_A(N_3, N_3) = \mathbb{F}_p$.

On the other hand, in the wildly ramified prime $p = 2$, we find $\dim(\mathrm{Ad}) = 0$, implying that $\mathrm{Ext}^1_A(N_3, N_3) = \mathbb{F}_2^2$.

7.3.4 The space $[\![\mathbb{Z}[\sqrt{d}]/\mathfrak{H}]\!]$

Theorem 7.2 *The space*

$$[\![\mathbb{Z}[\sqrt{d}]/\mathfrak{H}]\!] = \Big(\mathsf{Mod}\big(\mathbb{Z}[\sqrt{d}]/\mathfrak{H}\big), \mathfrak{O}\Big)$$

is a non-commutative thickening of $\mathbb{Z}$. *If* $M_{/\mathbb{F}_p}$ *is an unramified point*

$$\hat{\mathfrak{O}}_{\{M\}} = \mathrm{End}_{\mathbb{F}_p}(M).$$

In the tamely ramified case we have, with $\mathsf{N} := \{N_1, N_2, N_3\}$,

$$\hat{H}_{\mathsf{N}} = \begin{pmatrix} \mathbb{F}_p[[t_{11}]] & \langle t_{12}\rangle & \langle t_{13}\rangle \\ \langle t_{21}\rangle & \mathbb{F}_p[[t_{22}]] & \langle t_{23}\rangle \\ 0 & 0 & \mathbb{F}_p[[t_{33}]] \end{pmatrix}$$

In the wildly ramified case ($p = 2$), *we have*

$$\hat{H}_{\mathsf{N}} = \begin{pmatrix} \frac{\mathbb{F}_2\langle\langle u_1, u_2\rangle\rangle}{(u_1^2, u_2^2)} & \langle t_{12}^1, t_{12}^2\rangle & \langle t_{13}^1, t_{13}^2\rangle \\ \langle t_{21}^1, t_{21}^2\rangle & \frac{\mathbb{F}_2\langle\langle v_1, v_2\rangle\rangle}{(v_1^2, v_2^2)} & \langle t_{23}^1, t_{23}^2\rangle \\ 0 & 0 & \frac{\mathbb{F}_2\langle\langle w_1, w_2\rangle\rangle}{(w_1^2, w_2^2)} \end{pmatrix}$$

In both cases the versal family is

$$\hat{\mathfrak{O}}_{\mathsf{N}} = \mathrm{Hom}_{\mathbb{F}_p}(\mathsf{N}) \otimes_{\mathbb{F}_p} \hat{H}_{\mathsf{N}}.$$

Proof The only thing not proven in the discussion above are the obstructions in the wild ramification points. We omit this computation.

Finally, note that $\mathtt{ram}(\mathfrak{o}_F\langle\mathfrak{H}\rangle)$ is the set of ramified primes. □

7.4 Orders over a curve

Let $Y := \mathrm{Spec}\, R$, with $R := \bar{\mathfrak{o}}[u, v]/(f(u, v))$, be an arithmetic surface over $\bar{\mathfrak{o}}$ such that $\zeta = \zeta_3 \in \mathfrak{o}$ is a (primitive) third root of unity. Then

$$A := \frac{R\langle x, y\rangle}{(xy - \zeta yx, x^3 - u, y^3 - v)} = \frac{\bar{\mathfrak{o}}[u, v]\langle x, y\rangle}{\big(f(u, v), xy - \zeta yx, x^3 - u, y^3 - v\big)}$$

is an algebra over Y with central scheme $Y = \mathrm{Spec}\, \mathfrak{Z}(A)$ itself.

Let M' be the A-module $M' := k\boldsymbol{e}$ with actions

$$u\boldsymbol{e} = \alpha\boldsymbol{e}, \quad v\boldsymbol{e} = \beta\boldsymbol{e}, \quad x\boldsymbol{e} = a\boldsymbol{e}, \quad y\boldsymbol{e} = b\boldsymbol{e}, \quad \alpha, \beta, a, b \in k.$$

Note that $f(\alpha,\beta)=0$. This is an A-module over the closed point $(u-\alpha, v-\beta)$, $\alpha,\beta\in k=k(\mathfrak{p})$, where $\mathfrak{p}\in \operatorname{Spec}\mathfrak{o}$.

We have

$$\boldsymbol{u}\boldsymbol{e}=\alpha\boldsymbol{e},\;\; x\boldsymbol{e}=a\boldsymbol{e} \quad\Longrightarrow\quad x^3\boldsymbol{e}=u\boldsymbol{e}=\alpha\boldsymbol{e}$$

so $a^3=\alpha$. Therefore, there are three possibilities for a, namely, $a=\sqrt[3]{\alpha}$, $a_1:=\zeta\sqrt[3]{\alpha}=\zeta a$ and $a_2:=\zeta^2\sqrt[3]{\alpha}=\zeta^2 a$. The same applies to v, y, β and b.

For M' to be an A-module we need to have that $(xy-\zeta yx)\boldsymbol{e}=0$:

$$(xy-\zeta yx)\boldsymbol{e}=ab-\zeta ab=(1-\zeta)ab=0,$$

hence $ab=0$. Assume that $b=0$, implying that $\beta=0$.

Take two A-modules

$$M:=k\boldsymbol{e},\quad u\boldsymbol{e}=\alpha\boldsymbol{e},\;\; x\boldsymbol{e}=a\boldsymbol{e}$$

and

$$N:=k\boldsymbol{f},\quad u\boldsymbol{f}=\alpha\boldsymbol{f},\;\; x\boldsymbol{f}=\zeta a\boldsymbol{f}.$$

Let δ be a derivation $\delta: A\to \operatorname{Hom}(M,N)$. Put $\delta(x)\boldsymbol{e}:=d_x\boldsymbol{f}$, and, in addition $\delta(u)\boldsymbol{e}:=d_u\boldsymbol{f}$. We have

$$\begin{aligned}\delta(x^3-u)\boldsymbol{e}&=\big(\delta(x)x^2+x\delta(x)x+x^2\delta(x)-\delta(u)\big)\boldsymbol{e}\\&=(a^2d_x+\zeta a^2d_x+\zeta^2a^2d_x-d_u)\boldsymbol{f}\\&=(1+\zeta+\zeta^2)a^2d_x\boldsymbol{f}-d_u\boldsymbol{f}\\&=-d_u\boldsymbol{f}.\end{aligned}$$

Since ζ is a third root of unity $1+\zeta+\zeta^2=0$. Hence, $d_u=0$ and d_x can be chosen to be a free parameter.

Also, putting $\delta(v):=d_v$ and remembering that $u\boldsymbol{e}=\alpha\boldsymbol{e}$ and $v\boldsymbol{e}=0$,

$$\delta(uv-vu)\boldsymbol{e}=\big(\delta(u)v+u\delta(v)-\delta(v)u-v\delta(u)\big)\boldsymbol{e}=0.$$

Therefore, we can choose d_v as a free parameter.

Furthermore, we need to have $\delta(xy-\zeta yx)\boldsymbol{e}=0$ so

$$\delta(xy-\zeta yx)\boldsymbol{e}=\big(\delta(x)y+x\delta(y)-\zeta\delta(y)x-\zeta y\delta(x)\big)\boldsymbol{e}=\zeta ad_y\boldsymbol{f}-\zeta ad_y\boldsymbol{f}=0,$$

where $d_y:=\delta(y)$, can be chosen as a free parameter.

So far we have d_x, d_v and d_y as free parameters.

Let $\theta\in\operatorname{Hom}(M,N)$ with $\theta\boldsymbol{e}=t\boldsymbol{f}$. We directly see that $\dim(\mathrm{Ad})=1$ since

$$(\theta x-x\theta)\boldsymbol{e}=\theta x\boldsymbol{e}-x\theta\boldsymbol{e}=(t_1a-\zeta t_1a)\boldsymbol{f}=(1-\zeta)at_1\boldsymbol{f}.$$

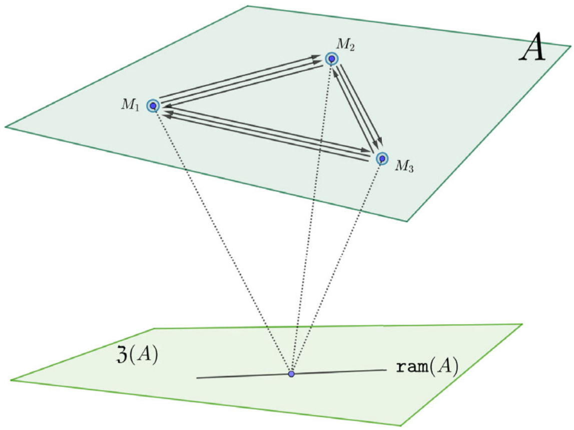
Therefore,

$$\operatorname{Ext}^1_A(M,N)=k^2.$$

On the other hand, choose N' as the module where x acts as $x\boldsymbol{e}=a_2\boldsymbol{e}$. We then get

$$\begin{aligned}\delta(x^3-u)\boldsymbol{e}&=\big(\delta(x)x^2+x\delta(x)x+x^2\delta(x)-\delta(u)\big)\boldsymbol{e}\\&=(a^2d_x+\zeta^2a^2d_x+\zeta^4a^2d_x-d_u)\boldsymbol{f}\\&=(1+\zeta^2+\zeta^4)a^2d_x\boldsymbol{f}-d_u\boldsymbol{f}\\&=-d_u\boldsymbol{f},\end{aligned}$$

Fig. 2 Tangent situation



since $\zeta^4 = \zeta$. Hence d_x is still free and $d_u = 0$. We also, still, have that d_v is free. However,

$$\begin{aligned}\delta(xy - \zeta yx)\boldsymbol{e} &= \big(\delta(x)y + x\delta(y) - \zeta\delta(y)x - \zeta y\delta(x)\big)\boldsymbol{e}\\ &= (a_2 d_y - \zeta a d_y)\boldsymbol{f}\\ &= \zeta(\zeta - 1)a d_y \boldsymbol{f}.\end{aligned}$$

Therefore, $d_y = 0$. The inner derivations are clearly still 1-dimensional. This means that

$$\mathrm{Ext}^1_A(M, N') = k.$$

Shifting the points cyclically we find that the diagram must look like the depiction in Fig. 2.

We easily find that

$$\mathrm{Ext}^1_A(M, M) = \begin{cases} 1, & \mathrm{char}(k) \neq 3\\ 2, & \mathrm{char}(k) = 3.\end{cases}$$

The adjacency matrix becomes when $\mathrm{char}(k) \neq 3$,

$$E = \begin{pmatrix} 1 & 2 & 1\\ 1 & 1 & 2\\ 2 & 1 & 1\end{pmatrix}$$

and, when $\mathrm{char}(k) = 3$,

$$E = \begin{pmatrix} 2 & 2 & 1\\ 1 & 2 & 2\\ 2 & 1 & 2\end{pmatrix}$$

In the first case the characteristic polynomial is $P(\lambda) = \lambda^3 - 3\lambda^2 - 3\lambda - 4$ and in the second $P(\lambda) = \lambda^3 - 6\lambda^2 - 6\lambda - 5$. Hence, we find that the non-commutative height is dependent on the characteristic of the ground field.

We leave for the reader to play around with rational points, divisors and intersection theory and report back to the author when finished.

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