

TWISTED DERIVATIONS, p -ADIC L -FUNCTIONS AND EXPLICIT RECIPROCITY LAWS

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Abstract

The explicit reciprocity law of Bloch–Kato (a later, conjectured, generalization was given by Perrin-Riou and subsequently proved by Colmez) is a fundamental result in the context of p -adic Hodge theory, p -adic Galois representations and L -functions of motives. In this paper we investigate possible relations between Bloch–Kato reciprocity and p -adic L -functions, on the one hand, and twisted derivations and (non-associative) algebraic structures that these form, on the other. We are able to show a close relationship between the Bloch–Kato law in the special case of Tate motives, and we construct, for general motives (in their crystalline representations), “Coleman maps” in the sense of recent papers by A. Lei, D. Loeffler and S.L. Zerbes, which might give new interesting p -adic L -functions when applied to Euler systems.

1 Introduction

This paper concerns explicit reciprocity laws, p -adic L -functions, and their (possible) relation to twisted derivations. Explicit reciprocity laws are often formulated as relations (or dualities) between arithmetic, geometric and analytic objects and generally provide, often conjecturally, deep information on the objects involved. The reciprocity law that will concern us here is the explicit law of Bloch–Kato [BK90] and Perrin-Riou [PR94] (which is a vast generalization of the reciprocity law in local class field theory). This law is expressed using p -adic Hodge theory and involves crystalline realizations of motives (essentially p -adic representations of Galois groups) and their p -adic L -functions.

The constructions concerning p -adic L -functions that follow, works for all motives (in their crystalline realizations) but due to the complexities involved (and the author’s limitations) the explicit reciprocity law is only discussed in the simplest case of the Tate motives $\mathbb{Q}(j)$.

The main underlying algebraic structure underlying this paper is a twisted version of a Lie algebra. In fact, we define a *hom-Lie algebra* L/A as a non-associative algebra over a (commutative) ring A , with product $\langle\langle \cdot, \cdot \rangle\rangle$, such that

- (i) $\langle\langle x, x \rangle\rangle = 0$, for $x \in L$, and
- (ii) $\odot_{x,y,z} \left(\langle\langle \sigma(x), \langle\langle y, z \rangle\rangle \rangle + q_\sigma \cdot \langle\langle x, \langle\langle y, z \rangle\rangle \rangle \right) = 0$.

Here σ is an endomorphism of L , $q_\sigma \in A$ and $\circlearrowright_{x,y,z}$ denotes cyclic permutation of x, y, z . See definition 2.2 for the precise definition. Notice that if $\sigma = \text{id}$ (this implies that $q_\sigma = 1$) we get the definition of a Lie algebra.

As a short historical account on the emergence of hom-Lie algebras, it is probably fair to say that it started, like many other things, with Euler and the study of “ q -analogues” of special functions. The primordial protagonist for this paper, namely the notion of q -difference (or q -differential) operator was considered by F.H. Jackson who studied these operators, and their inverses, “ q -integrals”, in the early twentieth century. To recall the classical setting, let σ be the automorphism $\sigma(t) = qt$ on $\mathbb{C}((t))$, where $q \in \mathbb{C}^\times$. Then the q -difference operator is defined as

$$\partial_q(f(t)) := \frac{\sigma(f) - f}{\sigma(t) - t} = \frac{f(qt) - f(t)}{(q - 1)t}.$$

In the limit $q \rightarrow 1$, we get the ordinary derivation. A short remark on the q -integral is found in remark 2.5 below.

Later, q -derivations were generalized and studied by ring-theorists in the guise of “ σ -derivations”, i.e., linear operators ∂_σ satisfying a twisted Leibniz rule, $\partial_\sigma(ab) = \partial_\sigma(a)b + \sigma(a)\partial_\sigma(b)$, especially in the context of so called Ore (after Ø. Ore) extensions of rings.

In addition, q -analogues of different objects turn up on a day-to-day basis in combinatorics, special functions, functions over finite fields, quantum groups (these are often Ore extensions, also known as skew polynomial rings), number theory and mathematical physics (“ q -deformations”) to name a few areas. However, it is only relatively recently that q -difference operators have begun to interest arithmetically inclined researchers. In the early years of the present century, Y. André [And01], L. di Vizio [DV02] and others started investigating q -calculus in the context of number theory and arithmetic geometry. In addition, q -derivations made a recent appearance in (integral) p -adic Hodge theory as a q -deformed de Rham complex in the announcement [BMS15].

The notion of a hom-Lie algebra was introduced in [HLS06] as an attempt to put an algebraic structure on spaces of σ -derivations, analogously to Lie algebras for ordinary derivations. However, at the time hom-Lie algebras were introduced we were unaware of any possible number-theoretic applications. Instead, the main motivating examples were q -deformation algebras appearing in physics (mostly from quantum field theory). One such example is the Virasoro algebra, which in the centreless version (also known as the Witt algebra) makes a disguised appearance later in this note. As indicated above, many of these were constructed using q -deformed derivations, i.e., q -difference operators, but a general underlying algebraic structure was lacking.

Observing (we were of course not the first ones to do this) that q -difference operators are special cases of so-called σ -twisted derivations (σ -derivations, σ -differential operators, . . .), with σ an endomorphism on some algebra underlying the structure (algebras of functions on some space), we defined a twisted multiplication of such σ -derivations. From the resulting structure we abstracted the notion of hom-Lie algebra (and later for more general σ -derivations, quasi-hom-Lie algebras and more generally quasi-Lie algebras). Later other people generalized this in different directions, but the notion originates as algebras of σ -twisted derivations.

The starting point of the present project was when I noticed the striking similarity between the structure constants of a certain type of hom-Lie algebras on the one hand, and Euler factors appearing in the explicit reciprocity laws of Bloch–Kato [BK90] and Perrin-Riou [PR94], on the other. I had the subsequent idea that it could be possible to “generate” explicit reciprocity laws as structure constants of hom-Lie algebras. Due to my limited abilities I didn’t get very far, but in this note I present some vague indication that something along these lines might be possible. I will leave this as a project to explore for someone more able than me.

The paper is organized as follows. First, in section 2 comes a short recap concerning twisted derivations and hom-Lie algebras. Here we only very briefly indicate the main points and refer to [Lar17] for full details. Section 3 is a short summary of the relevant material from the theory of (φ, Γ) -modules and p -adic Hodge theory.

In section 4 comes the first applications of hom-Lie algebras applied to (φ, Γ) -modules, and in particular, explicit reciprocity laws. In this section we first study the structure of the hom-Lie algebras coming from twisted derivations attached to the φ - and Γ -structures. In particular we prove that the hom-Lie algebras are generated, as left modules over certain (“period”) rings, by one element, and we compute the products explicitly. By observing that the structure constants appearing are exactly certain elements appearing in the Bloch–Kato reciprocity law for $\mathbb{Q}_p(j)$, we turn to a more detailed study of the relationship in section 5.

Finally, in section 6 we construct p -adic “ L -functions” from a hom-Lie algebra that is a p -deformation of the Lie algebra \mathfrak{sl}_2 . The first part of section 6 is devoted to the construction of this p -deformed \mathfrak{sl}_2 and studying weight modules of this hom-Lie algebra. Lastly, we use this algebra to generalize the construction of Coleman maps given in [LLZ10] and [LLZ11], which then, when applied to Euler systems, provide p -adic analytic functions that could be viewed as p -adic “ L -functions” of some motive. If these functions are interesting is a question that I leave for the future to judge.

One final remark: there are many open threads and unexplored possibilities in this paper. The reasons for not going further is primarily due to (1) the author’s abilities and (2) to not overstate the paper’s importance by letting it grow beyond any reasonable bound. Anyone interested in pursuing any idea is much encouraged to do so.

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2 Hom-Lie algebras and twisted derivations

All commutative rings will be assumed associative and unital.

2.1 Generalities

Let A be a commutative algebra over a commutative domain k , and let $\sigma : A \rightarrow A$ be a k -linear map on A . Then a *twisted derivation* (or σ -*derivation* as in the introduction) on A is a k -linear map $\partial : A \rightarrow A$ satisfying

$$\partial(ab) = \partial(a)b + \sigma(a)\partial(b).$$

We can generalize this as follows. Let A and σ be as above, and $M \in \text{ob}(\text{Mod}(A))$. The action of $a \in A$ on $m \in M$ will be denoted $a.m$. Then, a *twisted derivation on M* is k -linear map $\partial : M \rightarrow M$ such that

$$\partial(a.m) = \partial_A(a).m + \sigma(a).\partial(m), \quad (2.1)$$

where, by necessity, $\partial_A : A \rightarrow A$ is a twisted derivation on A (in the first sense). We call ∂_A the *restriction of ∂ to A* . Finally, a *twisted module derivation* is a k -linear map $\partial : A \rightarrow M$ such that

$$\partial(ab) = b.\partial(a) + \sigma(a).\partial(b),$$

for $\sigma \in \text{End}(A)$. Normally we will not differentiate between left and right module structures, but there are times when such a distinction would be necessary.

We will sometimes refer to the above as σ -*twisted (module) derivations* if we want to emphasize which σ we refer to.

Let $\sigma \in \text{End}(A)$ and denote by $\sigma^*A := A \otimes_{A,\sigma} A$, the extension of scalars along σ . This means that we consider A as a left module over itself via σ , i.e., $a.b := \sigma(a)b$. The right module structure is left unchanged. If M is an A -module, we put

$$\sigma^*M := (\sigma^*A) \otimes_A M = A \otimes_{A,\sigma} M,$$

i.e., M is endowed with left module structure $a.m := \sigma(a)m$, and once more, the right structure is unaffected. We now note that a σ -derivation d_σ on A is actually a derivation $d : A \rightarrow \sigma^*A$ and conversely. Indeed,

$$d(ab) = d(a)b + a.d(b) = d(a)b + \sigma(a)d(b).$$

In the same manner, a σ -derivation $d_\sigma : A \rightarrow M$ is a derivation $d : A \rightarrow \sigma^*M$, and conversely. Therefore, there is a one-to-one correspondence between σ -derivations $d_\sigma : A \rightarrow M$ and derivations $d : A \rightarrow \sigma^*M$.

Example 2.1 (The “universal” example). Let M be an A -module. Suppose $\sigma : M \rightarrow M$ is σ -semilinear, i.e., $\sigma(a.m) = \sigma(a).\sigma(m)$, for $a \in A$ and $m \in M$, where $\sigma \in \text{End}(A)$. Then, a small computation shows that for all $b \in A$, $\partial := b(\text{id} - \sigma) : M \rightarrow M$, is a σ -twisted derivation on M . Notice that if $M = A$, we automatically get $\sigma = \sigma$.

We will denote the (left) A -module of σ -twisted derivations on A (or M) as $\mathcal{D}_\sigma(A)$ (respectively, as $\mathcal{D}_\sigma(M)$).

2.2 Difference modules

Definition 2.1. A σ -*difference ring*, or σ -*ring* for short, is a ring A together with a $\sigma \in \text{End}(A)$; a σ -*difference module*, or σ -*module*, (M, σ) is a module over a difference ring (A, σ) together with a σ -linear endomorphism σ .

A σ -module can be viewed as a difference equation, in the exact same way as a differential module can be viewed as a differential equation. Now, let (E, σ) be a σ -difference module over a σ -difference ring (A, σ) . Put $\delta_\sigma := t(\text{id} - \sigma)$ for some $t \in A$.

Consider the map

$$\nabla^{(\sigma)} : E \rightarrow A \cdot \underline{\varepsilon} \otimes_A E, \quad m \mapsto \underline{\varepsilon} \otimes t(\text{id} - \sigma)(m), \quad (2.2)$$

where $A \cdot \underline{\varepsilon}$ is the canonical rank one A -module with basis $\underline{\varepsilon}$. This map satisfies a twisted Leibniz rule:

$$\begin{aligned} \nabla^{(\sigma)}(am) &= \underline{\varepsilon} \otimes (t(\text{id} - \sigma)(a)m) + \underline{\varepsilon} \otimes (a^\sigma t(\text{id} - \sigma)(m)) \\ &= (t(\text{id} - \sigma)(a) \cdot \underline{\varepsilon}) \otimes m + a^\sigma \cdot \underline{\varepsilon} \otimes t(\text{id} - \sigma)(m), \end{aligned}$$

i.e.,

$$\nabla^{(\sigma)}(am) = \partial_\sigma(a) \cdot \underline{\varepsilon} \otimes m + a^\sigma \nabla^{(\sigma)}m.$$

In addition, we can easily see that

$$\nabla^{(\sigma)} \circ \sigma = q \cdot \sigma \circ \nabla^{(\sigma)},$$

where $q = \sigma(t)/t$.

Lemma 2.1. Keeping the notation from above, there is a canonical σ -twisted connection $\nabla_M^{(\sigma)}$ given by (2.2). Conversely, localizing at t if necessary, given a σ -twisted connection we have a canonical σ -difference module (M_∇, φ) as the kernel of $\nabla^{(\sigma)}$.

Lemma 2.2. Let $A \in \text{ob}(\text{Com}(k))$ be a σ -difference k -algebra and suppose there is an $x \in A$, such that $x - \sigma(x) \in A^\times$. Let, in addition, M be a σ -module M . Then any σ -twisted derivation on M is of the form

$$\Delta_\sigma = (x - \sigma(x))^{-1} \partial_A(x)(\text{id} - \sigma),$$

where ∂_A is the restriction of Δ_σ to A . Furthermore, if M is torsion-free over A , then $A \cdot \Delta_\sigma = \mathcal{D}_\sigma(M)$ is free of rank one.

Proof. See [Lar17, Lemma 2.4]. \square

Lemma 2.3. If A is a UFD, and $\sigma \in \text{End}(A)$, then

$$\Delta_\sigma := \frac{\text{id} - \sigma}{g}$$

generates $\text{Der}_\sigma(A)$ as a left A -module, where $g := \text{gcd}((\text{id} - \sigma)(A))$, the greatest common divisor of the set $(\text{id} - \sigma)(A)$.

Proof. This is a special case of [HLS06, Theorem 4]. \square

Notice that the first lemma and the second lemma say slightly different things. The second lemma states that g is a *factor* in $(\text{id} - \sigma)(a)$ for all $a \in A$, and can be cancelled.

2.3 Algebras of twisted derivations

Let M be an A -module. Recall that the k -module of σ -twisted derivations is denoted

$$\mathcal{D}_\sigma(M) := \left\{ \partial \in \text{End}_k(M) \mid \partial(a.m) = \partial_A(a).m + \sigma(a).\partial(m) \right\}$$

and is a left A -modules. The A -module $\mathcal{D}_\sigma(A)$ is certainly not a Lie algebra.

2.3.1 Equivariant hom-Lie algebras

Let G be a group, A a k -algebra and M an A -module.

Definition 2.2. An *equivariant hom-Lie algebra* for G over A is an $A[G]$ -module M together with a k -bilinear product $\langle\langle \cdot, \cdot \rangle\rangle$ on M such that

(hL1.) $\langle\langle a, a \rangle\rangle = 0$, for all $a \in M$;

(hL2.) $\odot_{a,b,c} (\langle\langle a^\sigma, \langle\langle b, c \rangle\rangle \rangle + q_\sigma \cdot \langle\langle a, \langle\langle b, c \rangle\rangle \rangle) = 0$, for all $\sigma \in G$ and some $q_\sigma \in A$.

A morphism of equivariant hom-Lie algebras (M, G) and (M', G') is a pair (f, ψ) of a morphism of k -modules $f : M \rightarrow M'$ and $\psi : G \rightarrow G'$ such that $f \circ \sigma = \psi(\sigma) \circ f$, and $f \langle\langle a, b \rangle\rangle_M = \langle\langle f(a), f(b) \rangle\rangle_{M'}$.

Notice that the definition implies that for a morphism

$$(f, \psi) : (M, G) \rightarrow (M', G')$$

we must have $f(q_\sigma) = q_{\psi(\sigma)}$.

We get a *hom-Lie algebra* when we simply consider one $\sigma \in G$. Notice that for $\sigma = \text{id}$ we get a Lie algebra.

Remark 2.1. Actually, one can make sense of the proposition that hom-Lie algebras are Lie algebras “in a suitably twisted category” (see [CG11]).

2.3.2 The hom-Lie algebra structure on $\mathcal{D}_\sigma(M)$

Let, as before, $A \in \text{ob}(\text{Com}(k))$ and let $\sigma \in \text{End}(A)$. Denote by δ_σ a σ -twisted derivation on M whose restriction to A is ∂ , i.e., $\delta_\sigma \in \mathcal{D}_\sigma(M)$ and $\partial \in \mathcal{D}_\sigma(A)$. Assume that $\sigma(\text{Ann}(\delta_\sigma)) \subseteq \text{Ann}(\delta_\sigma)$, where

$$\text{Ann}(\delta_\sigma) := \{a \in A \mid a\delta_\sigma(m) = 0, \quad \text{for all } m \in M\},$$

and that

$$\partial \circ \sigma = q \cdot \sigma \circ \partial, \quad \text{for some } q \in A. \quad (2.3)$$

Form the left A -module

$$A \cdot \delta_\sigma := \{a \cdot \delta_\sigma \mid a \in A\}.$$

Define

$$\langle\langle a \cdot \delta_\sigma, b \cdot \delta_\sigma \rangle\rangle := \sigma(a) \cdot \delta_\sigma(b \cdot \delta_\sigma) - \sigma(b) \cdot \delta_\sigma(a \cdot \delta_\sigma). \quad (2.4)$$

This should be interpreted as

$$\langle\langle a \cdot \delta_\sigma, b \cdot \delta_\sigma \rangle\rangle(m) := \sigma(a) \cdot \delta_\sigma(b \cdot \delta_\sigma(m)) - \sigma(b) \cdot \delta_\sigma(a \cdot \delta_\sigma(m)), \quad (2.5)$$

for $m \in M$. We now have the following fundamental theorem.

Theorem 2.4. Under the above assumptions, equation (2.4) gives a well-defined k -linear product on $A \cdot \delta_\sigma$ such that

- (i) $\langle\langle a \cdot \delta_\sigma, b \cdot \delta_\sigma \rangle\rangle = (\sigma(a)\partial(b) - \sigma(b)\partial(a)) \cdot \delta_\sigma;$
- (ii) $\langle\langle a \cdot \delta_\sigma, a \cdot \delta_\sigma \rangle\rangle = 0;$
- (iii) $\circlearrowleft_{a,b,c} (\langle\langle \sigma(a) \cdot \delta_\sigma, \langle\langle b \cdot \delta_\sigma, c \cdot \delta_\sigma \rangle\rangle \rangle + q \cdot \langle\langle a \cdot \delta_\sigma, \langle\langle b \cdot \delta_\sigma, c \cdot \delta_\sigma \rangle\rangle \rangle) = 0,$

where, in (iii), q is the same as in (2.3).

Corollary 2.5. The A -module $A \cdot \delta_\sigma$ is a hom-Lie algebra.

For proofs of these statements, see [HLS06].

2.4 The Jackson \mathfrak{sl}_2

For the rest of the paper, we always assume that $\text{char}(k) \neq 2$, where k is a prime field. Most of the constructions go through without this assumption, but this would require different sets of generators and some more work.

Let R be any commutative k -algebra (with $\text{char}(k) \neq 2$), and $B := R[t, t^{-1}]$ the ring of Laurent polynomials over R . The Witt Lie algebra W is the Lie algebra of derivations of B . It is a classical fact that

$$W := \text{Der}_R(B) = B \cdot t \frac{d}{dt}$$

and it is easily checked that the Lie brackets are given as

$$[e_i, e_j] = (j - i)e_{i+j}, \quad \text{with } e_i := t^{i+1} \frac{d}{dt}, \quad i \in \mathbb{Z}.$$

Observe that there is a copy of $\mathfrak{sl}_2(R)$ inside W spanned by $\{e_{-1}, -2e_0, -e_1\}$. It is customary to put $e := e_{-1}$, $h := -2e_0$ and $f := -e_1$. Obviously we then get

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

The element h spans the Cartan subalgebra of \mathfrak{sl}_2 .

Remark 2.2. In the classical Lie-theoretic context, $R = \mathbb{C}$ and W is geometrically the vector fields on the unit circle. However, this was not the case Witt himself studied. His interest was the case $B = \mathbb{F}_q[t_1, t_2, \dots, t_n]/(t_1^q, t_2^q, \dots, t_n^q)$, his interest stemming from finite group theory.

Fix $\sigma \in \text{End}_k(B)$. Since we are only interested here in the simplest possible case we will assume that $\sigma(t) = qt$, for some $q \in R^\times$ and $\sigma|_R = \text{id}$. We will now deform W in the “ σ -direction” in the sense that we replace $t \frac{d}{dt}$ by

$$\nabla^{(\sigma)} := ((1 - q)t)^{-1}(\text{id} - \sigma).$$

In more well-known parlance, we will “ q -deform” W .

Define the q -number $[n]_q$ to be

$$[n]_q := 1 + q + q^2 + \dots + q^{n-1} = \frac{1 - q^n}{1 - q}, \quad q \neq 1.$$

Notice that for $n > 0$, $[-n]_q = -q^{-n}[n]_q$. Then, putting $e_i := t^{i+1}\nabla^{(\sigma)}$, a simple calculation shows that

$$\langle\langle e_i, e_j \rangle\rangle = ([j]_q - [i]_q)e_{i+j}.$$

We denote the resulting hom-Lie algebra W_q . It can be beneficial to observe that, if $q = \zeta_n$ is an n -th root of unity, then $W_{\mathbb{Z}/n} := \{W_{q^a} \mid 0 \leq a \leq n-1\}$ is an equivariant hom-Lie algebra over \mathbb{Z}/n .

A small calculation shows that the set

$$\{\mathbf{e}_\sigma := e_{-1}, \quad \mathbf{h}_\sigma := -2e_0, \quad \mathbf{f}_\sigma := -e_1\}, \quad (2.6)$$

generates a sub-hom-Lie algebra, the *Jackson* \mathfrak{sl}_2 , in W_q with products

$$\langle\langle \mathbf{h}_\sigma, \mathbf{e}_\sigma \rangle\rangle = 2\mathbf{e}_\sigma, \quad \langle\langle \mathbf{h}_\sigma, \mathbf{f}_\sigma \rangle\rangle = -2q\mathbf{f}_\sigma, \quad \langle\langle \mathbf{e}_\sigma, \mathbf{f}_\sigma \rangle\rangle = \frac{q+1}{2}\mathbf{h}_\sigma. \quad (2.7)$$

Notice that $q = 1$ gives back the above presentation of \mathfrak{sl}_2 . We put

$$\mathbb{J}_q := \left(\text{Span}_R(\{\mathbf{e}_\sigma, \mathbf{h}_\sigma, \mathbf{f}_\sigma\}), (2.7) \right).$$

The name “Jackson \mathfrak{sl}_2 ” for the q -deformed \mathfrak{sl}_2 given by (2.7) was introduced in [LS07]. In this note we denote it by \mathbb{J}_q . It is rather easy to convince oneself that \mathbb{J}_q is simple (as hom-Lie algebra) for all $q \neq 0$. Obviously

$$\mathbb{J}_q^- := \text{Span}_R\{\mathbf{e}_\sigma, \mathbf{h}_\sigma\} \quad \text{and} \quad \mathbb{J}_q^+ = \text{Span}_R\{\mathbf{h}_\sigma, \mathbf{f}_\sigma\}$$

are sub-hom-Lie algebras of \mathbb{J}_q ; in fact, they are *Lie* algebras. As above we note that

$$\mathbb{J}_{\mathbb{Z}/n} := \{\mathbb{J}_{q^a} \mid 0 \leq a \leq n-1\}$$

is an equivariant hom-Lie algebra over \mathbb{Z}/n , whenever q is an n -th root of unity.

Remark 2.3. Let L be a hom-Lie algebra. It would obviously be very interesting to classify all sub-hom-Lie algebras inside L that are actually Lie algebras.

Observe that \mathbb{J}_q is a sub-hom-Lie algebra of W_q . The construction given in [LS07] is slightly different but the result is the same. Also, we originally constructed it only over fields of characteristic zero, but the same applies for any commutative ring of characteristic not 2 (in which case one has to take another set of generators).

Remark 2.4. A natural question is of course: when are two \mathbb{J}_{q_1} and \mathbb{J}_{q_2} , $q_1 \neq q_2$, isomorphic? This seems to be a rather difficult question involving determining K -rational points on a certain algebraic variety.

Remark 2.5. Associated to the σ -derivation $\nabla^{(\sigma)}$, there is an inverse called the *Jackson σ -integral*, defined formally by inverting $\text{id} - \sigma$:

$$\int f d_\sigma t := (1 - q) \sum_{i=0}^{\infty} t \sigma^i(f) = (1 - q) \sum_{i=0}^{\infty} t f(q^i t).$$

It is natural to wonder if this has any uses in p -adic integration (for instance for p -adic measures) and p -adic Hodge theory.

2.5 Weight modules

Exactly as for the Lie algebra \mathfrak{sl}_2 one can define the notion of weight modules for \mathbb{J}_q . It is important to note that the same construction can be generalized to the whole W_q .

As (2.4) indicates, the correct definition of a hom-Lie module is as follows. Let L be a hom-Lie algebra over a ring B , with product $\langle\langle \cdot, \cdot \rangle\rangle$. Then a hom-Lie algebra module M is a B -module with an action of L such that

$$\langle\langle a, b \rangle\rangle(m) = \sigma(a) \cdot (b \cdot m) - \sigma(b) \cdot (a \cdot m), \quad a, b \in L. \quad (2.8)$$

Proposition 2.6. Suppose that $L = B \cdot \Delta_\sigma$, for Δ_σ a σ -derivation on B . The endomorphism σ is extended to L by $\sigma(b \cdot \Delta_\sigma) := \sigma(b) \cdot \Delta_\sigma$. If M is a σ -module over (B, σ) , then M is a hom-Lie algebra module over L under (2.8).

Proof. This follows directly from the above definitions, together with (2.5). \square

As a consequence,

Corollary 2.7. A difference equation (M, σ) over a difference ring (B, σ) defines a canonical hom-Lie algebra module.

Let now M be a vector space over a \mathbb{Q} -algebra R with an action of \mathbb{J}_q . Observe that we do not need to assume that M is a hom-Lie module over \mathbb{J}_q at this point. Assume that $v \in M$ is an eigenvector of \mathbf{h}_σ with eigenvalue $\xi = \xi(v) \in R$. The eigenvalue ξ is called a *weight* for M and

$$M_\xi := \{w \in M \mid \mathbf{h}_\sigma \cdot w = \xi w\}$$

the *weight space of weight* ξ in M .

We put

$$v_{-1} := 0, \quad v_k := \frac{1}{[k]_q!} \mathbf{f}_\sigma^k v, \quad k \geq 0, \quad \text{with } v_0 = v.$$

The linear span $M_v := \bigoplus_{k=0}^{\infty} Rv_k$ is called a *weight module of weight* ξ for \mathbb{J}_q .

Proposition 2.8. We have

$$\begin{aligned} \mathbf{f}_\sigma \cdot v_k &= [k+1]_q v_{k+1}, \\ \mathbf{h}_\sigma \cdot v_k &= (\xi(v)q^k - 2[k]_q) v_k, \\ \mathbf{e}_\sigma \cdot v_k &= \left(\frac{\xi(v)q^{k-1}(q+1)}{2} - [k-1]_q \right) v_{k-1}. \end{aligned}$$

Notice that when $q = 1$ we get the ordinary weight theory for \mathfrak{sl}_2 .

Proof. The proof consists of straightforward computations using induction and (2.7) together with (2.8), but it is somewhat hard to attack it correctly, so we include some hints. Suppose we are interested in computing $\mathbf{h}_\sigma(v_1)$ (the case v_0 is clear by definition). Notice that $v_1 = \mathbf{f}_\sigma v_0$. Then we should have

$$\sigma(\mathbf{h}_\sigma)(\mathbf{f}_\sigma(v_0)) - \sigma(\mathbf{f}_\sigma)(\mathbf{h}_\sigma(v_0)) = \langle\langle \mathbf{h}_\sigma, \mathbf{f}_\sigma \rangle\rangle = -2q\mathbf{f}_\sigma(v_0),$$

whence,

$$\sigma(\mathbf{h}_\sigma)(\mathbf{f}_\sigma(v_0)) = \sigma(\mathbf{f}_\sigma)(\mathbf{h}_\sigma(v_0)) - 2q\mathbf{f}_\sigma(v_0).$$

By definition (yes, this is the “correct” definition!)

$$\sigma(\mathbf{e}_\sigma) = \mathbf{e}_\sigma, \quad \sigma(\mathbf{h}_\sigma) = q\mathbf{h}_\sigma, \quad \sigma(\mathbf{f}_\sigma) = q^2\mathbf{f}_\sigma,$$

so, rewriting the above we get

$$\mathbf{h}_\sigma(v_1) = \mathbf{h}_\sigma(\mathbf{f}_\sigma(v_0)) = q\mathbf{f}_\sigma(\mathbf{h}_\sigma(v_0)) - 2v_1.$$

Since $\mathbf{h}_\sigma(v_0) = \xi(v)v_0$, we get

$$\mathbf{h}_\sigma(v_1) = \xi(v_0)q\mathbf{f}_\sigma(v_0) - 2v_1 = (\xi(v_0)q - 2)v_1.$$

The rest is by induction and the other cases follows similarly. \square

The standard examples are $M = R[t]$, or its finite-dimensional truncations $M_n = R[t]/(t^n)$, with $v_0 = -\frac{\xi}{2}t$, for $\xi \in R$, and with \mathbb{J}_q given its natural presentation as q -differential operators. This gives weight modules of weight ξ .

More generally, let M be any σ -module over $(R[t], \sigma)$. Then we get a natural weight module for \mathbb{J}_q . Notice that if M is of finite rank as an R -module and q is not a root of unity, then (possibly up to a localization) the \mathbf{h}_σ -eigenvectors v_k are linearly independent over R and so generate a finite rank R -submodule of M that is also a \mathbb{J}_q -module.

3 (φ, Γ_∞) -modules

3.1 Rings of periods and (φ, Γ_∞) -modules

We need some basics from p -adic Hodge theory. The following recollection will be superficial at best. The detailed constructions of the rings that will follow are rather complicated and we refer to, e.g., [Ber04a], and the references given therein, for more details.

In the following we let K_0 be a finite unramified extension of \mathbb{Q}_p and let K be a finite totally ramified extension of K_0 . Hence, if k is the residue class field of K then $K_0 = W(k)[p^{-1}]$. We denote the canonical Frobenius morphism on K_0 , by σ_{K_0} .

Let K^a denote an algebraic closure of K . Throughout V will be a p -adic representation of $G_K := \text{Gal}(K^a/K)$, i.e., a \mathbb{Q}_p -vector space with a continuous action of G_K .

Put $K_\infty := \bigcup_{n \geq 0} K(\mu_{p^n})$, where $K(\mu_{p^n})$ denotes the field, generated over K , by the p^n -th roots of unity in K^a . We have the following diagram of extensions

$$\begin{array}{ccccccc} & & & G_K & & & \\ & & & \curvearrowright & & & \\ K_0 & \longrightarrow & K & & K_\infty & \longrightarrow & K^a & \longrightarrow & \widehat{K^a} = \mathbb{C}_p \\ & & & \Gamma_\infty & & & H_K & & \end{array}$$

Hence $H_K = \text{Gal}(K^a/K_\infty)$ and $\Gamma_\infty = G_K/H_K$. The group Γ_∞ decomposes as $\Gamma_\infty = \mathfrak{D} \times \Gamma$, where $\mathfrak{D} := \text{Gal}(K(\mu_p)/K)$ and $\Gamma = \text{Gal}(K_\infty/K(\mu_p))$. We will normally consider only the case where $K = K_0$. The *Iwasawa algebra* of Γ_∞ is

$$\Lambda(\Gamma_\infty) = \mathbb{Z}_p[\mathfrak{D}][[1 - \gamma]],$$

for some topological generator γ of Γ . For L/\mathbb{Q}_p a finite extension, we can extend this as

$$\Lambda_{\mathfrak{o}_L}(\Gamma_\infty) = \mathfrak{o}_L[\mathfrak{D}] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[1 - \gamma]], \quad \text{and} \quad \Lambda_L(\Gamma_\infty) = \Lambda_{\mathfrak{o}_L}(\Gamma_\infty) \otimes_{\mathfrak{o}_L} L.$$

We also introduce the distribution algebra

$$\mathcal{H}_L(\Gamma_\infty) := \{f \in L[\mathfrak{D}][[1 - \gamma]] \mid f \text{ converges on } D^\circ\},$$

where D° is the open unit disc in \mathbb{C}_p . We can identify $\Lambda_L(\Gamma_\infty)$ with the elements in $\mathcal{H}_L(\Gamma_\infty)$ that are bounded on D° .

We will from now assume that we are given a \mathbb{Q}_p -algebra \mathbf{B} with a continuous action of G_K and a Frobenius morphism φ . Let $\omega \in \mathbf{B}$ be a distinguished element, which in the cases we will consider will be either the element $t := \log([\epsilon])$, or the element $\pi := [\epsilon] - 1$. Here $[\epsilon]$ is the Teichmüller lift of a coherent sequence of p^n -th roots of unity in a certain canonical characteristic p ring. We refer to [Ber02] or [Ber04a] for the details.

For any (topological) \mathbb{Q}_p -algebra \mathbf{S} , we say that V is \mathbf{S} -admissible if $\mathbf{S} \otimes_{\mathbb{Q}_p} V = \mathbf{S}^d$ as $\mathbf{S}[G_K]$ -modules. Here $d := \dim_{\mathbb{Q}_p}(V)$. Observe that it is implicit here that \mathbf{S} has a continuous G_K -action. Normally, \mathbf{S} will also have a distinguished “Frobenius” map φ , i.e., a \mathbb{Q}_p -linear map $\varphi : \mathbf{S} \rightarrow \mathbf{S}$.

Let V be a p -adic representation which is \mathbf{S} -admissible. Associated to every \mathbf{S} -admissible representation is a Dieudonné module defined as

$$D_{\mathbf{S}}(V) := (\mathbf{S} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

When \mathbf{S} has a σ_{K_0} -linear Frobenius φ , $D_{\mathbf{S}}(V)$ becomes a φ -module over \mathbf{S}^{G_K} .

Typical examples of rings \mathbf{S} are the period rings \mathbf{B}_{cris} , \mathbf{B}_{st} , \mathbf{A} and \mathbf{B} of Fontaine, the “plus” rings \mathbf{A}^+ , \mathbf{B}^+ and $\mathbf{B}_{\text{rig}}^+$, as well as the “dagged” rings \mathbf{B}^\dagger and $\mathbf{B}_{\text{rig}}^\dagger$ of Berger, Colmez and others.

The only difference between the \mathbf{A} - and \mathbf{B} -rings is that in the \mathbf{B} -variants, p is invertible, while in the \mathbf{A} -variants this is not the case; in fact, often $\mathbf{B} = \mathbf{A}[1/p]$. In some cases we will need to invert the element ω (for instance, $\omega = \pi$ is not invertible in plus-rings, and $\omega = t$ is not invertible in dagged rings).

Put $\mathbf{S}_K := \mathbf{S}^{H_K}$. Note that this ring has an induced action of Γ_∞ . An \mathbf{S}_K -vector space M with a semilinear action of φ , commuting with a semilinear action of Γ_∞ , is called a (φ, Γ_∞) -module; if the canonical

$$\varphi^* M := \mathbf{S}_K \otimes_{\mathbf{S}_K, \varphi} M \rightarrow M$$

is an isomorphism, M is an *étale* (φ, Γ_∞) -module. This last is equivalent to M having a φ -stable basis. For a p -adic representation V , the Dieudonné module

$$D_{\mathbf{S}, K}(V) := (\mathbf{S} \otimes_{\mathbb{Q}_p} V)^{H_K}$$

is a (φ, Γ_∞) -module over \mathbf{S}_K .

To simplify the discussion below we will assume from now on that $K = K_0$. All the constructions below go through without this assumption, but there is no immediate pay-off for the resulting notational inconvenience that thereby ensues.

We define the rings \mathbf{A}_K^+ , \mathbf{B}_K^+ , \mathbf{A}_K and \mathbf{B}_K as

$$\mathbf{A}_K^+ := \mathfrak{o}_K[[\pi]], \quad \mathbf{B}_K^+ := \mathbf{A}_K^+[p^{-1}] = \mathbf{A}_K^+ \otimes_{\mathfrak{o}_K} K$$

and

$$\mathbf{A}_K := \widehat{\mathfrak{o}_K[[\pi]]}[\pi^{-1}]_{\hat{p}}, \quad \mathbf{B}_K := \mathbf{A}_K[p^{-1}],$$

where the subscript \hat{p} means that the completion is taken with respect to p . Observe that $\mathbf{B}_K^+ \neq K[[\pi]]$ (as one might, as I was for a long time, be tempted to think).

There are canonical injections $\mathbf{A}_K^+ \hookrightarrow \mathbf{A}^+$ and $\mathbf{B}_K^+ \hookrightarrow \mathbf{B}^+$. Furthermore, let \mathbf{B} be the p -adic completion of $\mathbf{B}_K^{\text{unr}}$ inside a certain ring $\tilde{\mathbf{B}}$ that won't be defined here.

One can prove that the local field extension $\mathbf{B}/\varphi(\mathbf{B})$ is of degree p and so one can define the important averaging (or trace) operator

$$\psi : \mathbf{B} \rightarrow \mathbf{B}, \quad \psi(x) := p^{-1}\varphi^{-1}\text{Tr}_{\mathbf{B}/\varphi(\mathbf{B})}(x).$$

This operator commutes with G_K and can be extended to all period rings \mathbf{S} and \mathbf{S}_K that we will consider. Extending ψ to the first factor of $\mathbf{S} \otimes V$, we see that it can also be extended to all relevant Dieudonné modules. Essentially by construction one has

$$\psi(\varphi(x)a) = x\psi(a), \quad \psi(x\varphi(a)) = \psi(x)a, \quad x \in \mathbf{S}, a \in \mathbf{D}_{\mathbf{S}}(V),$$

in particular $\psi(\varphi(x)) = x$.

Since we assume that $K = K_0$ we can give the action of φ and γ on π explicitly as

$$\varphi(\pi) = (1 + \pi)^p - 1, \quad \text{and} \quad \gamma(\pi) = (1 + \pi)^{\chi(\gamma)} - 1.$$

In addition, we can assume that $\chi(\gamma) = 1 + p^n \in 1 + p^n\mathbb{Z}_p = U^{(n)}(\mathbb{Z}_p)$, for some $n \in \mathbb{N}$.

The ring \mathbf{A}_K is a discrete valuation ring, hence a principal ideal domain (PID), and therefore a unique factorization domain (UFD). The same applies to \mathbf{B}_K^+ . Since \mathfrak{o}_K is the ring of integers in a local field, it is a regular PID, hence a regular UFD, and so $\mathfrak{o}_K[[\pi]]$ is also UFD. The ring \mathbf{B}_K is even a field. Finally, the ring $\mathbf{B}_{\text{rig},K}^+$ is defined as

$$\mathbf{B}_{\text{rig},K}^+ := \left\{ f(\pi) \left| f(T) = \sum_{i \geq 0} f_i T^i, f_i \in K, \text{converging on } D^\circ \subset \mathbb{C}_p \right. \right\},$$

where D° is the open unit disc. The associated Dieudonné module is defined as

$$V \mapsto \mathbf{D}_{\text{rig}}^+(V) := \mathbf{B}_{\text{rig},K}^+ \otimes_{\mathbf{B}_K^+} \mathbf{D}^+(V).$$

Notice that we haven't actually defined \mathbf{D}^+ , referring instead to e.g., [Ber02] for details.

It is possible to recover $\mathbf{D}_{\text{cris}}(V)$ knowing $\mathbf{D}_{\text{rig}}^+(V)$. In fact, a theorem of Berger states that

$$\mathbf{D}_{\text{cris}}(V) = \left(\mathbf{D}_{\text{rig}}^+(V)[t^{-1}] \right)^{\Gamma_\infty},$$

and, in case V is positive (see 6.3 for what this means), we get the stronger statement

$$\mathbf{D}_{\text{cris}}(V) = \left(\mathbf{D}_{\text{rig}}^+(V) \right)^{\Gamma_\infty}.$$

4 p -adic hom-Lie algebras and (φ, Γ_∞) -modules

We denote by χ the cyclotomic character $\chi : G_K \rightarrow \mathbb{Z}_p^\times$. The main virtue of the chosen element $t \in \mathbf{S}$ is that it satisfies (see [Ber02, p. 222] for instance)

$$\varphi(t) = pt \in \mathbf{S} \quad \text{and} \quad \gamma(t) = \chi(\gamma)t \in \mathbf{S}_K, \quad \gamma \in \Gamma_\infty.$$

From now on, V will denote a p -adic representation of G_K over \mathbb{Q}_p . The dimension over \mathbb{Q}_p will be denoted by d . The Tate twist of V is as usual defined as

$$V(r) := V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r), \quad \text{where} \quad \mathbb{Q}_p(r) := \mathbb{Q}_p \otimes_{\mathbb{Q}_p} \underline{\varepsilon}^{\otimes r},$$

where G_K acts on $\underline{\varepsilon}$ through Γ_∞ via the cyclotomic character by $\gamma(\underline{\varepsilon}) := \chi(\gamma)\underline{\varepsilon}$ and extended to $\mathbb{Q}_p(r)$ as

$$\gamma(a \otimes \underline{\varepsilon}^{\otimes r}) := \chi(\gamma)^r a \otimes \underline{\varepsilon}^{\otimes r}, \quad a \in \mathbb{Q}_p.$$

We extend the action of φ to $\underline{\varepsilon}$ by

$$\varphi(\underline{\varepsilon}^{\otimes r}) := p^r \underline{\varepsilon}^{\otimes r}, \quad r \in \mathbb{Z}.$$

This implies that

$$\mathbf{D}_{\mathbf{S},K}(V(r)) = \mathbf{D}_{\mathbf{S},K}(V) \otimes \underline{\varepsilon}^{\otimes r}, \quad \text{with} \quad \varphi_{\mathbf{D}_{\mathbf{S},K}(V(r))} = p^{-r} \varphi_{\mathbf{D}_{\mathbf{S},K}(V)}$$

in order to compensate for the extra factor p^r that appears (the φ -structures are invariant under Tate twists on the φ -module side; only the Γ_∞ -structure is affected by the Tate twists).

Example 4.1. Important special cases of the above are:

$$\begin{aligned} \mathbf{D}(\mathbb{Q}_p(r)) &= \mathbf{D}(\mathbb{Q}_p) \otimes \underline{\varepsilon}^{\otimes r} = (\mathbf{S} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p)^G \otimes \underline{\varepsilon}^{\otimes r} \\ &= \begin{cases} K_0 \otimes \underline{\varepsilon}^{\otimes r} = K_0(r), & \text{if } \mathbf{D} = \mathbf{D}_{\text{cris}}, \text{ with } G = G_K \\ \mathbf{S}_K \otimes \underline{\varepsilon}^{\otimes r} = \mathbf{S}_K(r), & \text{if } \mathbf{D} = \mathbf{D}_{\mathbf{S},K}, \text{ with } G = H_K. \end{cases} \end{aligned}$$

Observe that in the crystalline case, there is no action of G_K and in particular no action of Γ_∞ .

What follows now is a sequence of theorems corresponding to different situations of interest. The proofs are, if not word-for-word identical, easily adapted to the proof of the first instant of the statement (which in turn follows directly from previously proven statements). Not all of these results are used later, but simply recorded for completeness.

4.1 φ -rings

First, observe that even if p is not invertible, $1 - p^r$ is (for $r \geq 1$), since p is in the maximal ideal of \mathcal{O}_{K_0} . Therefore, since \mathbf{S} is a K_0 -algebra, $1 - p^r$ is invertible in \mathbf{S} .

Put

$$\partial := (1 - p)^{-1}(\text{id} - \varphi) : \mathbf{S} \rightarrow \mathbf{S}.$$

Clearly $\partial(t) = t$ and $\partial \circ \varphi = \varphi \circ \partial$. Therefore, $\mathbf{S} \cdot \partial \subseteq \mathcal{D}_\varphi(\mathbf{S})$ comes equipped with a canonical hom-Lie algebra structure by theorem 2.4, with $q = 1$.

From now on, adjoin to \mathbf{S} , if necessary, the inverse of t . In order to avoid excessively heavy notation we still denote this ring \mathbf{S} . Under this assumption, we put

$$\nabla^{(\varphi)} := t^{-1}\partial = ((1-p)t)^{-1}(\text{id} - \varphi).$$

We have the following theorem.

Theorem 4.1. The left \mathbf{S} -module $\mathcal{D}_\varphi(\mathbf{S})$ is free of rank one over with generator $\nabla^{(\varphi)}$. We also have

$$\nabla^{(\varphi)} \circ \varphi = p \cdot \varphi \circ \nabla^{(\varphi)},$$

and as a consequence $\mathcal{D}_\varphi(\mathbf{S})$ can be endowed with a canonical hom-Lie algebra structure.

Proof. The theorem follows from a straightforward computation and an application of lemma 2.2 and theorem 2.4 with $x = t$ and $\sigma = \varphi$. \square

Remark 4.1. If $t^{-1} \notin \mathbf{S}$, then there is no a priori reason why $\mathcal{D}_\varphi(\mathbf{S})$ should be free of rank one since there may be no elements $y \in \mathbf{S}$ such that $(\text{id} - \varphi)(y) \in \mathbf{S}^\times$.

4.2 φ -modules

Let \mathbf{D} be any φ -module over \mathbf{S}_K . Suppose first that $t \notin \mathbf{S}_K$. The primary interest under this assumption is the case $\mathbf{S} = \mathbf{B}_{\text{cris}}$, when $\mathbf{S}_K = K_0$. Then, as before, $\mathcal{D}_\varphi(\mathbf{D})$ and $\mathcal{D}_\varphi(\mathbf{S}_K)$ can be identified as hom-Lie algebras. We cannot in general claim that this module is free of rank one, for the same reason as in remark 4.1 (but with t instead of t^{-1} , of course). On the other hand we still have that the submodule $\mathbf{S}_K \cdot \partial$ comes with a hom-Lie algebra structure with $q = 1$.

In the particular case of primary interest to us, namely, when $\mathbf{S} = \mathbf{B}_{\text{cris}}$, we have that $\mathbf{S}_K = K_0$. This is a field and so in fact

$$\mathcal{D}_\varphi(\mathbf{D}) = \mathcal{D}_\varphi(K_0) = K_0(\text{id} - \varphi).$$

The same applies to any isocrystal over K_0 .

Put

$$\nabla_a^{(\varphi)} := a(\text{id} - \varphi), \quad a \in \mathbf{S}_K.$$

This $\nabla_a^{(\varphi)}$ is a \mathbb{Q}_p -linear operator on \mathbf{D} such that

$$\nabla_a^{(\varphi)}(bv) = (a(\text{id} - \varphi)(b))v + \varphi(b)\nabla_a^{(\varphi)}(v), \quad b \in \mathbf{S}_K, v \in \mathbf{D}.$$

Therefore, \mathbf{D} becomes a left $\mathcal{D}_\varphi(\mathbf{S}_K)$ -module, in addition to $\mathcal{D}_\varphi(\mathbf{D})$ making sense as a module over \mathbf{S}_K .

Assuming $t^{-1} \in \mathbf{S}_K$ and replacing \mathbf{S} in theorem 4.1 with \mathbf{S}_K , automatically yields (with $a^{-1} = (1-p)t$)

Theorem 4.2. Let \mathbf{D} be a φ -module over \mathbf{S}_K . Then the left \mathbf{S}_K -module $\mathcal{D}_\varphi(\mathbf{D})$ is free of rank one over with generator $\nabla^{(\varphi)}$. In addition,

$$\nabla^{(\varphi)} \circ \varphi = p \cdot \varphi \circ \nabla^{(\varphi)}.$$

Therefore, $\mathcal{D}_\varphi(\mathbf{D})$ can be endowed with a canonical hom-Lie algebra structure, naturally identified with the hom-Lie algebra structure on $\mathcal{D}_\varphi(\mathbf{S}_K)$.

Remark 4.2. The operator $\nabla^{(\varphi)}$ is almost a “monodromy operator”, so \mathbf{D} is nearly a (φ, N) -module with $N = \nabla^{(\varphi)}$. The problem causing it to not achieve this status, is that $\nabla^{(\varphi)}$ is not K_0 -linear.

4.3 (φ, Γ_∞) -modules

Of course, the φ -action is the same as above, so only the Γ_∞ -action is new (but essentially obvious in light of the previous theorems).

We have that $\gamma(t) = \chi(\gamma)t$ on \mathbf{S}_K . Suppose that $(1 - \chi(\gamma))t \in \mathbf{S}_K^\times$ and put

$$\nabla^{(\gamma)} := ((1 - \chi(\gamma))t)^{-1}(\text{id} - \gamma).$$

Observe that, since $\chi(\gamma) \in 1 + p^n\mathbb{Z}_p$ for some $n \in \mathbf{N}$, $1 - \chi(\gamma)^s \in p\mathbb{Z}_p$, so is only invertible in the rings where p is. This excludes the \mathbf{A} -rings.

Remark 4.3. The module $\mathcal{D}_\gamma(\mathbf{D})$ is a first approximation of the Lie algebra $\text{Lie}(\Gamma_\infty)$ acting on \mathbf{D} . In fact, the operator $\nabla^{(\gamma)}$ is, up to a factor, a first-order truncation of Sen’s differential operator Θ as given in, for instance, [Ber02]. This operator can be seen to generate the action of $\text{Lie}(\Gamma_\infty)$ on \mathbf{D} . This shows that \mathbf{D} is actually a difference equation (in the guise of a difference module) over \mathbf{S}_K .

Theorem 4.3. Let \mathbf{D} be a (φ, Γ_∞) -module over \mathbf{S}_K . Then the \mathbf{S}_K -module $\mathcal{D}_\gamma(\mathbf{D})$ is free of rank one, generated by the element $\nabla^{(\gamma)}$. We have

$$\nabla^{(\gamma)} \circ \gamma = \chi(\gamma) \cdot \gamma \circ \nabla^{(\gamma)},$$

so $\mathcal{D}_\gamma(\mathbf{D})$ can be endowed with a canonical hom-Lie algebra structure.

We note the following easy proposition.

Proposition 4.4. The above operators satisfy the relations

$$\nabla^{(\gamma)} \circ \varphi = p \cdot \varphi \circ \nabla^{(\gamma)}, \quad \nabla^{(\varphi)} \circ \gamma = \chi(\gamma) \cdot \gamma \circ \nabla^{(\varphi)},$$

and

$$pt \cdot \nabla^{(\varphi)} \circ \nabla^{(\gamma)} = \nabla^{(\gamma)} \circ (p - \varphi), \quad t \cdot \nabla^{(\gamma)} \circ \nabla^{(\varphi)} = \nabla^{(\varphi)} \circ (1 - \chi(\gamma)\gamma).$$

Proof. Simple computations. \square

For the next theorem we denote by \mathbf{S}' either of \mathbf{S} or \mathbf{S}_K .

Theorem 4.5. Assume that \mathbf{S}' is a UFD and put $g_\sigma := \gcd((\text{id} - \sigma)(\mathbf{S}'))$, where $\sigma = \varphi$ or γ . Let \mathbf{D} be a φ - or (φ, Γ_∞) -module over \mathbf{S}' . Then $\mathcal{D}_\sigma(\mathbf{D})$ is free of rank one over \mathbf{S}' with generator

$$\Delta^{(\sigma)} := \frac{\text{id} - \sigma}{g_\sigma}.$$

Furthermore,

$$\Delta^{(\sigma)} \circ \sigma = \frac{\sigma(g_\sigma)}{g_\sigma} \cdot \sigma \circ \Delta^{(\sigma)}$$

and so $\mathcal{D}_\sigma(\mathbf{D})$ is a hom-Lie algebra over \mathbf{S}' .

Proof. The first statement follows from lemma 2.3. The commutation relation is easily proved, noting that if g_σ divides a , then $\sigma(g_\sigma)$ divides $\sigma(a)$ and g_σ divides $\sigma(g_\sigma)$ (by definition of greatest common divisor). \square

Observe first that the statement of the theorem is independent of t . In addition, notice that g need not be invertible for this statement. A greatest common divisor is only well-defined up to units, but it is easy to see that any other g will give the same module and an isomorphic hom-Lie algebra structure.

4.4 Explicit hom-Lie products

We will now compute hom-Lie products.

4.4.1 Hom-Lie products with $\mathcal{S} = \mathcal{B}_{\text{cris}}$

First of all we note the anomalous case of $\mathcal{S} = \mathcal{B}_{\text{cris}}$ (applicable to any isocrystal). Hence, let V be any crystalline representation. Then $\mathcal{D}_{\text{cris}}(V)$ is a φ -module over $K_0 = (\mathcal{B}_{\text{cris}})^{G_K}$. We have that

$$\mathcal{D}_\varphi(K_0) = K_0(\text{id} - \varphi) = K_0 \cdot \partial,$$

with products

$$\langle\langle a \cdot \partial, b \cdot \partial \rangle\rangle = (\varphi(a)b - \varphi(b)a) \cdot \partial, \quad a, b \in K_0.$$

Obviously, we here have $\varphi = \varphi_0 = \varphi_{K_0}$. A φ -derivation on $\mathcal{D}_{\text{cris}}(V)$ is on the form

$$\partial_{\text{cris}} : \mathcal{D}_{\text{cris}}(V) \rightarrow \mathcal{D}_{\text{cris}}(V); \quad \partial_{\text{cris}} := a(\text{id} - \varphi_{\mathcal{D}_{\text{cris}}(V)}).$$

Recall that, we can identify $\mathcal{D}_\varphi(\mathcal{D}_{\text{cris}}(V))$ and $\mathcal{D}_\varphi(K_0)$ as hom-Lie algebras, but *not* as modules of operators (since $\mathcal{D}_\varphi(\mathcal{D}_{\text{cris}}(V))$ has another Frobenius).

Taking Tate twists we have $\mathcal{D}_{\text{cris}}(V(r)) = \mathcal{D}_{\text{cris}}(V) \otimes \underline{\varepsilon}^{\otimes r}$. This is still a φ -module over K_0 but now

$$\partial_{\text{cris}} = a(\text{id} - \varphi_{\mathcal{D}_{\text{cris}}(V(r))}) = a(\text{id} - p^{-r} \varphi_{\mathcal{D}_{\text{cris}}(V)})$$

and

$$\mathcal{D}_{\varphi_{\mathcal{D}_{\text{cris}}(V(r))}}(\mathcal{D}_{\text{cris}}(V(r))) = \mathcal{D}_{p^{-r} \varphi_{\mathcal{D}_{\text{cris}}(V)}}(\mathcal{D}_{\text{cris}}(V) \otimes \underline{\varepsilon}^{\otimes r}).$$

From the above we see that it is important to distinguish between the space $\mathcal{D}_\varphi(\mathcal{D}_{\text{cris}}(V))$ as an algebra, and as a module of operators.

4.4.2 Hom-Lie products over (φ, Γ_∞) -rings where t is invertible

Let \mathcal{D} denote a (φ, Γ_∞) -module over \mathcal{S}_K . When writing σ below, we mean either φ or γ , and q is then either p or $\chi(\gamma)$, depending on which σ we use.

Recall that $\nabla^{(\sigma)} = ((1 - q)t)^{-1}(\text{id} - \sigma)$ and that $\mathcal{D}_\sigma(\mathcal{S}_K)$ and $\mathcal{D}_\sigma(\mathcal{D})$ are generated by $\nabla^{(\sigma)}$ as free rank-one \mathcal{S}_K -module. Put

$$\delta_\sigma^i := t^i \nabla^{(\sigma)}, \quad i \in \mathbb{Z}.$$

Note that $\delta_\sigma^0 = \nabla^{(\sigma)}$. Clearly, δ_σ^i acts on \mathcal{D} for all $i \in \mathbb{Z}$, so \mathcal{D} is a $\mathcal{D}_\sigma(\mathcal{S}_K)$ -module (from the left), and also $\delta_\sigma^i \in \mathcal{D}_\sigma(\mathcal{D})$. Every $D \in \mathcal{D}_\sigma(\mathcal{S}_K)$ and $\mathcal{D}_\sigma(\mathcal{D})$ can be written (uniquely) as $D = a\delta_\sigma^i$, for some $a \in \mathcal{S}_K$ and $i \in \mathbb{Z}$.

Remark 4.4. We see that $\partial = \delta^1$ and this element generates the sub-hom-Lie algebra $\mathbf{S}_K \cdot \partial \subseteq \mathcal{D}_\sigma(\mathbf{D})$. I don't know if this is in fact the whole $\mathcal{D}_\sigma(\mathbf{D})$. Since in all cases we consider the ring \mathbf{S}_K is a domain, so the \mathbf{S}_K -module $\mathbf{S}_K \cdot \partial$ is at least free of rank one.

We note the formula

$$\delta_\sigma^i(t^n) = [n]_q t^{n-1+i}, \quad i, n \in \mathbb{Z}.$$

With this notation we can give the hom-Lie products explicitly, using theorem 2.4 (and a small computation, using the above formulas), as:

$$\langle\langle a\delta_\sigma^i, b\delta_\sigma^j \rangle\rangle = \beta \left(q^i \sigma(a)b - q^j \sigma(b)a \right) \delta_\sigma^{i+j-1}, \quad a, b \in \mathbf{S}_K, \quad (4.1)$$

where we have put $\beta := (1-q)^{-1}$. For notational convenience we introduce the operator $\hat{\delta} := \beta^{-1} \delta_\sigma^0$. Notice, for instance, that

$$\langle\langle \hat{\delta}, b\delta_\sigma^j \rangle\rangle = (b - p^j \varphi(b)) \delta_\sigma^{j-1},$$

and in particular the case when $b = 1$:

$$\langle\langle \hat{\delta}, \delta_\sigma^j \rangle\rangle = (1 - p^j) \delta_\sigma^{j-1}.$$

Introducing the projection operator

$$(-)^b : \mathcal{D}_\sigma(\mathbf{S}_K) \longrightarrow \mathbf{S}_K, \quad a\delta_\sigma^i \mapsto a$$

we see that we can produce Euler factors

$$\langle\langle \hat{\delta}, b\delta_\sigma^j \rangle\rangle^b = b - p^j \varphi(b),$$

and in particular then,

$$\langle\langle \hat{\delta}, \delta_\sigma^j \rangle\rangle^b = 1 - p^j.$$

Extending this to operators we can construct the Euler factor operators

$$\langle\langle \hat{\delta}, (-)\delta_\sigma^j \rangle\rangle^b = \text{id} - p^j \varphi.$$

These will prove useful later.

4.4.3 Hom-Lie products over (φ, Γ_∞) -rings where t is not invertible

Now, if t is not invertible in \mathbf{S}_K , we have three options. Either: (1) we look at the subalgebra $K[t] \subseteq \mathbf{S}_K$, or (2) we adjoin t^{-1} , or (3) we use the generator π instead of t .

Looking at the first case (1), we notice that $K[t]$ is a unique factorization domain so, putting

$$\Delta^{(\sigma)} := \frac{\text{id} - \sigma}{g_\sigma}, \quad g_\sigma := \gcd((\text{id} - \sigma)K[t]) = (1 - q)t,$$

we have that $\mathcal{D}_\sigma(K[t])$ is generated as a left $K[t]$ -module by $\nabla^{(\sigma)}$. With these definitions, the arguments from the previous subsection goes through word-for-word, with the same result (4.1).

In case (2), we use theorem 4.5 to construct the generator of $\mathcal{D}_\sigma(\mathbf{D})$ and the induced hom-Lie algebra structure. Unfortunately, due to the more complicated actions of φ and γ on π , the products in case (3) become too complicated to write out in full generality.

Remark 4.5. Using the above products one can consider \mathbb{Q}_p -linear maps as follows. First of all,

$$\langle\langle a\delta_*^i, - \rangle\rangle : \mathcal{D}_*(\mathcal{S}_K) \rightarrow \mathcal{D}_*(\mathcal{S}_K)$$

and

$$\langle\langle a\delta_*^i, (-)\delta_*^j \rangle\rangle : \mathcal{S}_K \rightarrow \mathcal{D}_*(\mathcal{S}_K)$$

can clearly be seen as \mathbb{Q}_p -linear operators. Further, fixing $m \in D$, we can construct the operator

$$\langle\langle a\delta_*^i, (-)\delta_*^j \rangle\rangle(m) : \mathcal{S}_K \rightarrow D.$$

These operators (and variants of them) can be useful to consider at times.

4.5 The hom-Lie algebra $\mathcal{D}_\sigma(\mathcal{B}^\sigma[t, t^{-1}])$

In this section, we let \mathcal{B}^σ denote either $\mathcal{B}_{\text{cris}}^{\varphi=1}$, $\mathcal{S}_K^{\varphi=1}$ or $K = \mathcal{S}_K^{\Gamma_\infty}$, depending on whether we use φ or γ . The ring $\mathcal{B}_{\text{cris}}^{\varphi=1}$ is a rather complicated ring, often denoted \mathcal{B}_e (see [Ber08]).

We will in this section assume that t is invertible. Recall the operators $\nabla^{(\sigma)} = ((1-q)t)^{-1}(\text{id} - \sigma)$. Recall also that it is not always true that $1-q$ is invertible in \mathcal{B}^σ when $q = \chi(\gamma)$. Observe that $\mathcal{B}^\sigma[t, t^{-1}]$ is a subalgebra of \mathcal{S}_K .

We put

$$W_\sigma := \mathcal{B}^\sigma[t, t^{-1}] \cdot \nabla^{(\sigma)} = \mathcal{B}^\sigma[t, t^{-1}] \cdot \delta_\sigma^0 = \mathcal{D}_\sigma(\mathcal{B}^\sigma[t, t^{-1}]).$$

From (4.1) we find

$$\langle\langle \delta_\sigma^i, \delta_\sigma^j \rangle\rangle = ([i]_q - [j]_q) \delta_\sigma^{i+j-1}, \quad i, j \in \mathbb{Z}. \quad (4.2)$$

This is a hom-Lie algebra called the *q-deformed Witt algebra* (or simply *q-Witt algebra*) in analogy with the Lie algebra case $q = 1$, which is the Lie Witt algebra (although *not* the algebra Witt himself studied). It is rather easy to see that W_σ is a simple hom-Lie algebra.

Remark 4.6. Actually, the relations for the *q*-Witt algebra in [HLS06] are \mathbb{Z} -graded. This is achieved by using a different set of generators (shifted in degree by one). However, in what follows it is more convenient to use the above generators.

When q is not a root of unity and \mathcal{B}^σ a field, there is a unique central extension of W_σ called the *q-Virasoro algebra* (in analogy with the Lie case $q = 1$). I expect that when q is an n th root of unity, there are n non-equivalent central extensions but I have never checked this (but it certainly should be done!). For more details, see [HLS06].

5 The case $V = \mathbb{Q}_p(r)$ and Bloch–Kato reciprocity

In this section we will look at the special case where $V = \mathbb{Q}_p(r)$, and a crystalline G_K -representation. We will not be able to be very detailed concerning some of the constructions that follow, but we will refer the reader to the appropriate places in the literature.

Put

$$K_n := K(\mu_n), \quad U_\infty := \varprojlim U_n, \quad B_\infty := \varprojlim K_n^\times,$$

where $U_n := U(K_n)$ are the principal units in K_n , and the projective limits are taken with respect to norm maps.

In [BK90] Bloch and Kato defined an “exponential map” from a Galois-theoretic object to a certain cohomology group. Namely,

$$\exp_V : \mathbb{D}_{\text{dR}}(V)/\text{Fil}^0 \mathbb{D}_{\text{dR}}(V) \longrightarrow H_e^1(K, V),$$

induced from a connecting homomorphism in cohomology. When V is de Rham (in particular if V is crystalline) \exp_V is actually an isomorphism [BK90, Theorem 4.1(ii)].

In the case $V = \mathbb{Q}_p(r)$,

$$\mathbb{D}_{\text{dR}}(\mathbb{Q}_p(r))/\text{Fil}^0 \mathbb{D}_{\text{dR}}(\mathbb{Q}_p(r)) \simeq K_0(r)$$

and

$$H_e^1(K, \mathbb{Q}_p(r)) \simeq \text{Hom}_\Gamma(U_\infty, \mathbb{Q}_p(r)),$$

so

$$\exp_{\mathbb{Q}_p(r)} : K_0(r) \rightarrow \text{Hom}_\Gamma(U_\infty, \mathbb{Q}_p(r)).$$

By the comment above, $\exp_{\mathbb{Q}_p(r)}$ is an isomorphism (unless $r = 0$, in which case the right-hand side is trivial).

There is another connecting homomorphism of importance. This is induced from the exact sequence [BK90, (1.13)]

$$0 \longrightarrow \mathbb{Q}_p(r) \longrightarrow \text{Fil}^r \mathbf{B}_{\text{cris}}^+ \xrightarrow{\text{id} - p^{-r}\varphi} \mathbf{B}_{\text{cris}}^+ \longrightarrow 0.$$

Indeed, taking G_K -invariants gives a morphism

$$\underline{\partial}^r : K = H^0(K, \mathbf{B}_{\text{cris}}^+) \longrightarrow H^1(K, \mathbb{Q}_p(r)) = \text{Hom}_\Gamma(U_\infty, \mathbb{Q}_p(r)).$$

These two homomorphisms $\underline{\partial}^r$ and $\exp_{\mathbb{Q}_p(r)}$ are connected via the equality [Sai15, Theorem 4.5.7]

$$\exp_{\mathbb{Q}_p(r)}(a)(\tau) = -\underline{\partial}^r((\text{id} - p^{-r}\varphi)(a))(\tau), \quad a \in K \text{ and } \tau \in G_K. \quad (5.1)$$

In addition, the explicit reciprocity law of Bloch–Kato ([BK90, Theorem 2.1] with sign corrected in [dS95] and [Sai15, Theorem 4.1.1]) is given as

$$\underline{\partial}^r(a) = -\frac{1}{(r-1)!} \text{Tr}_{K/\mathbb{Q}_p}(a \cdot \Phi_{\text{CW}}^r(-)), \quad (5.2)$$

where

$$\Phi_{\text{CW}}^r : U_\infty \longrightarrow K$$

is the r -th Coates–Wiles homomorphism (see e.g., [Sai15, 4.4.2]). Combining (5.1) and (5.2), we write the reciprocity law as

$$\exp_{\mathbb{Q}_p(r)}(a) = -\frac{1}{(r-1)!} \text{Tr}_{K/\mathbb{Q}_p}(a \cdot \Phi_{\text{CW}}^r(-)). \quad (5.3)$$

Define

$$\exp_{\mathbb{Q}_p(r)}^b(a)(\tau) := -\underline{\partial}^r (b(\text{id} - p^{-r}\varphi)(a))(\tau), \quad b \in K.$$

This defines a group homomorphism (of additive groups)

$$\begin{aligned} \underline{E} : \mathcal{D}_\varphi(K(r)) &\longrightarrow \text{Hom}(K(r), \text{Hom}_\Gamma(U_\infty, \mathbb{Q}_p(r))) \\ b(\text{id} - p^{-r}\varphi) &\longmapsto \exp_{\mathbb{Q}_p(r)}^b. \end{aligned}$$

Now, the question is: is it possible to generalize the Coates–Wiles homomorphism to some $\tilde{\Phi}_{\text{CW}}^r$ so that we can construct families of reciprocity laws,

$$\exp_{\mathbb{Q}_p(r)}^b(a) = -\frac{1}{(r-1)!} \text{Tr}_{K/\mathbb{Q}_p} \left(a \cdot \tilde{\Phi}_{\text{CW}}^r(-) \right),$$

parametrized by $b \in K$ via \underline{E} . I have a feeling that this should be possible in some form. A natural follow-up question would then be if it is possible to extend to representations coming from other motives than $\mathbb{Q}_p(r)$.

The generalized exponential map \exp^b can be packaged into a larger object. For this, recall the Euler factor operator

$$(\text{id} - p^{-r}\varphi) = \langle\langle \hat{\delta}, (-)\delta_\varphi^{-r} \rangle\rangle^b.$$

Instead of \mathbf{B}_{cris} we use $\mathbf{B}_{\text{rig},K}^+$. Define the pairing

$$\underline{\text{Exp}} : \mathcal{D}_\varphi(\mathbf{B}_{\text{rig},K}^+[t^{-1}]) \times \mathcal{D}_\varphi(\mathbf{B}_{\text{rig},K}^+[t^{-1}]) \longrightarrow \prod_{k \in \mathbb{Z}} \text{Hom}_\Gamma(U_\infty, \mathbb{Q}_p(k))$$

defined by

$$(a\delta^i, b\delta^j) \longmapsto -\underline{\partial}^{i+j} \left(b \langle\langle \hat{\delta}, a\delta^{i-j} \rangle\rangle^b \right) \in \text{Hom}_\Gamma(U_\infty, \mathbb{Q}_p(i+j)).$$

Fixing $a \in \mathbf{B}_{\text{rig},K}^+$ and i this pairing specializes to

$$\begin{aligned} \underline{\text{Exp}}_a^i : \mathcal{D}_\varphi(\mathbf{B}_{\text{rig},K}^+[t^{-1}]) &\longrightarrow \prod_{j \in \mathbb{Z}} \text{Hom}(K, \text{Hom}_\Gamma(U_\infty, \mathbb{Q}_p(i+j))) \\ b\delta_\varphi^j &\longmapsto -\underline{\partial}^{i+j} \left(b \langle\langle \hat{\delta}, a(-)\delta_\varphi^{i-j} \rangle\rangle^b \right). \end{aligned}$$

The case $i = 0$ is especially interesting:

$$\begin{aligned} \underline{\text{Exp}}_a &:= \underline{\text{Exp}}_a^0 : \mathcal{D}_\varphi(\mathbf{B}_{\text{rig},K}^+[t^{-1}]) \longrightarrow \prod_{j \in \mathbb{Z}} \text{Hom}(K, \text{Hom}_\Gamma(U_\infty, \mathbb{Q}_p(j))) \\ b\delta_\varphi^j &\longmapsto -\underline{\partial}^j \left(b \langle\langle \hat{\delta}, a(-)\delta_\varphi^{-j} \rangle\rangle^b \right). \end{aligned}$$

This \underline{E}_a defines a group morphism in the b -argument and, if $a = 1$, we get the previous morphism \underline{E} .

We can approach this from another angle, emulating the constructions in [Ber03] (which are reformulations of the main theorems of [PR94] in terms of (φ, Γ) -modules). This will be even more sketchy than the above and I will assume that the reader is to a large extent well acquainted with the constructions therein.

The idea of Perrin-Riou is to extend the exponential map to a infinite family of exponential maps over the whole tower $K \subset K_\infty$, namely,

$$\exp_{n,V} : K_n \otimes_K \mathbf{D}_{\text{cris}}(V) \longrightarrow H^1(K_n, V)$$

and then glueing these together mapping into the Iwasawa cohomology group $H_{\text{Iw}}^1(K, V) = \varprojlim H^1(K_n, V)$ (the limit taken with respect to corestriction of cohomology groups). Here we only consider the 0-th level $K = K_0$.

Still assuming that $V = \mathbb{Q}_p(r)$ is crystalline, we now look at the ring $\mathbf{S}_K = \mathbf{B}_{\text{rig},K}^+[t^{-1}]$. In [Ber03, Prop. 1.8] is constructed a group morphism

$$h_{K,V}^1 : \mathbf{D}_{\text{rig},K}^+(V)^{\psi=1} \longrightarrow H^1(K, V)$$

(we will discuss the map ψ later), which in the case $V = \mathbb{Q}_p(r)$ reduces to

$$h_{K,V}^1 : \mathbf{B}_{\text{rig},K}^+(r)^{\psi=1} \longrightarrow H^1(K, \mathbb{Q}_p(r)) = \text{Hom}_\Gamma(U_\infty, \mathbb{Q}_p(r)).$$

Put

$$\tilde{\nabla}_i := \delta_\gamma^1 - [i]_{\chi(\gamma)}.$$

One easily checks that

$$\tilde{\nabla}_0 \left(\mathbf{B}_{\text{rig},K}^+ \right) \subset t \mathbf{B}_{\text{rig},K}^+, \quad \text{and so} \quad \tilde{\nabla}_{i-1} \circ \tilde{\nabla}_{i-2} \circ \cdots \circ \tilde{\nabla}_0 \left(\mathbf{B}_{\text{rig},K}^+ \right) \subset t^i \mathbf{B}_{\text{rig},K}^+.$$

Then the same argument as in the beginning of [Ber03, II.2] is applicable and so if $y \in \mathbf{B}_{\text{rig},K}^+ \otimes \mathbf{D}_{\text{cris}}(\mathbb{Q}_p(r)) = \mathbf{B}_{\text{rig},K}^+(r)$, then

$$\tilde{\nabla}_{i-1} \circ \tilde{\nabla}_{i-2} \circ \cdots \circ \tilde{\nabla}_0(y) \in \mathbf{D}_{\text{rig},K}^+(\mathbb{Q}_p(r))^{\psi=1} = \mathbf{B}_{\text{rig},K}^+(r)^{\psi=1}.$$

Applying $h_{K,\mathbb{Q}_p(r)}^1$ to this element gives

$$\Phi_y := h_{K,\mathbb{Q}_p(r)}^1 \left(\tilde{\nabla}_{i-1} \circ \cdots \circ \tilde{\nabla}_0(y) \right) \in H^1(K, \mathbb{Q}_p(r)) = \text{Hom}_\Gamma(U_\infty, \mathbb{Q}_p(r)).$$

This element Φ_y , “should” be the the generalized $\tilde{\Phi}_{\text{CW}}^r$, for some suitably chosen y .

6 Coleman maps and p -adic L -functions

We will in this section look at the cases when \mathbf{S}_K is one of $\mathbf{A}_K^+ = \mathfrak{o}_K[[\pi]]$, $\mathbf{B}_K^+ = \mathbf{A}_K^+[1/p] = \mathfrak{o}_K[[\pi]][1/p]$ or $\mathbf{B}_{\text{rig},K}^+$. One should view \mathbf{B}_K^+ as power series converging, and bounded, on the open unit disc D° , and $\mathbf{B}_{\text{rig},K}^+$ as those only converging on D° . There are thus natural inclusions

$$\mathbf{A}_K^+ \subset \mathbf{B}_K^+ \subset \mathbf{B}_{\text{rig},K}^+.$$

Recall from the end of section 3.1 that we defined

$$\mathbf{D}_{\text{rig}}^+(V) := \mathbf{B}_{\text{rig},K}^+ \otimes_{\mathbf{B}_K^+} \mathbf{D}^+(V),$$

subsequently remarked that

$$\mathbf{D}_{\text{cris}}(V) = \left(\mathbf{D}_{\text{rig}}^+(V)[t^{-1}] \right)^{\Gamma_\infty} \quad \text{and} \quad \mathbf{D}_{\text{cris}}(V) = \left(\mathbf{D}_{\text{rig}}^+(V) \right)^{\Gamma_\infty},$$

the last equality holding when V is positive. In addition,

$$\mathbf{B}_{\text{rig},K}^+[t^{-1}] \otimes_K \mathbf{D}_{\text{cris}}(V) = \mathbf{B}_{\text{rig},K}^+[t^{-1}] \otimes_K \mathbf{D}_{\text{rig}}^+(V) = \mathbf{D}_{\text{rig}}^+(V)[t^{-1}],$$

by [Ber04b, II.3].

Let $\mathbf{B}^\sigma := (\mathbf{B}_{\text{rig},K}^+)^{\varphi=1}$ and recall the notation $\nabla^{(\sigma)} = ((1-q)t)^{-1}(1-\sigma)$, where $\sigma = \varphi$ or γ . We have the following inclusions:

$$\begin{array}{ccc} \mathbf{B}^\sigma[t, t^{-1}] \cdot \nabla^{(\sigma)} & \xrightarrow{=} & \mathcal{D}_\sigma(\mathbf{B}^\sigma[t, t^{-1}]) \\ \downarrow & & \downarrow \\ \mathbf{B}^\sigma[[t]][t^{-1}] \cdot \nabla^{(\sigma)} & \xrightarrow{=} & \mathcal{D}_\sigma(\mathbf{B}^\sigma[[t]][t^{-1}]) \\ \downarrow & & \downarrow \\ \mathbf{S}_K \cdot \nabla^{(\sigma)} & \xrightarrow{=} & \mathcal{D}_\sigma(\mathbf{S}_K). \end{array}$$

Hence, if \mathbf{D} is a (φ, Γ_∞) -module over \mathbf{S}_K , then we see that $\mathcal{D}_\sigma(\mathbf{B}^\sigma[t, t^{-1}])$ acts on $\mathcal{D}_\sigma(\mathbf{D})$.

6.1 The φ -case

Recall the elements

$$\mathbf{e}_\sigma := \nabla^{(\sigma)} = \delta_\sigma^0, \quad \mathbf{h}_\sigma := -2t\nabla^{(\sigma)} = -2\delta_\sigma^1, \quad \mathbf{f}_\sigma := -t^2\nabla^{(\sigma)} = -\delta_\sigma^2,$$

defined in (2.6), where $\delta_\sigma^i := t^i\nabla^{(\sigma)}$. In this section we look at the case when $\sigma = \varphi$.

Now, inside \mathbf{S}_K we consider the subalgebra $\mathbf{B}^\sigma[t, t^{-1}]$ and form \mathbb{J}_p :

$$\langle\langle \mathbf{h}_\varphi, \mathbf{e}_\varphi \rangle\rangle = 2\mathbf{e}_\varphi, \quad \langle\langle \mathbf{h}_\varphi, \mathbf{f}_\varphi \rangle\rangle = -2p\mathbf{f}_\varphi, \quad \langle\langle \mathbf{e}_\varphi, \mathbf{f}_\varphi \rangle\rangle = \frac{p+1}{2}\mathbf{h}_\varphi.$$

We immediately observe:

Proposition 6.1. The reduction of \mathbb{J}_p modulo p is the solvable *Lie* algebra

$$\langle\langle \mathbf{h}_\varphi, \mathbf{e}_\varphi \rangle\rangle = 2\mathbf{e}_\varphi, \quad \langle\langle \mathbf{h}_\varphi, \mathbf{f}_\varphi \rangle\rangle = 0, \quad \langle\langle \mathbf{e}_\varphi, \mathbf{f}_\varphi \rangle\rangle = \mathbf{h}_\varphi,$$

over \mathbf{B}^σ/p .

We have made the scaling $\mathbf{f}_\varphi \rightarrow \frac{1}{2}\mathbf{f}_\varphi$ in the proposition. Observe that the reduction is not $\mathfrak{sl}_2(\mathbb{F}_p)$. Hence we see (the known fact) that there are flat deformations (lifts) of Lie algebras over \mathbb{F}_p to \mathbb{Q}_p , that are not Lie algebras.

We will often view \mathbb{J}_p in the canonical representation as φ -derivations. In this representation one can see that \mathbb{J}_p is invariant under the action of the Iwasawa algebra.

Recall that $\beta = (1-p)^{-1}$ and that we assume $K = K_0$. In what follows we will use the φ -module $\mathbf{D}_{\text{cris}}(V)$ for illustration, but any other φ -module \mathbf{D} could equally well have been used.

Assume that there is an eigenvector v to φ in $\mathbf{D}_{\text{cris}}(V)$ with eigenvalue λ_v . Clearly, v is also an eigenvector to $\text{id} - \varphi$ with eigenvalue $1 - \lambda_v$ and to $\mathbf{h}_\varphi = -2\beta(1 - \varphi)$ with eigenvalue $-2\beta(1 - \lambda_v)$.

Remark 6.1. We might need to extend scalars here to include the eigenvalues. We work linearly over any such extension. For instance, if F is an extension of \mathbb{Q}_p , then φ is F -linear on $F \otimes_{\mathbb{Q}_p} \mathbf{D}_{\text{cris}}(V)$, i.e., $\varphi(a \otimes bv) = a \otimes \varphi(b)\varphi(v)$, for $a \in F$, $b \in K$ and $v \in \mathbf{D}_{\text{cris}}(V)$. In view of this, we will suppress any needed extension in the notation.

Put $u := t^i \otimes v \in \mathbf{B}_{\text{rig},K}^+[t^{-1}] \otimes_K \mathbf{D}_{\text{cris}}(V)$, for any $i \in \mathbb{Z}$. This is now also an eigenvector to \mathbf{h}_φ with eigenvalue $\xi(u) := -2\beta(1 - p^i \lambda_v)$. We now put

$$u_k := \frac{1}{[k]_p!} \mathbf{f}_\varphi^k u, \quad k \in \mathbb{Z}_{\geq 0}, \quad u_0 = u, \quad u_{-1} = 0$$

and define $\mathbf{e}_\varphi(u) = 0$. Observe that $u_k \in \mathbf{B}_{\text{rig},K}^+[t^{-1}] \otimes \mathbf{D}_{\text{cris}}(V)$, for all $k \geq -1$.

Proposition 6.2. The F -span of $\{u_k\}$ defines a \mathbb{J}_p -weight module

$$M_p(\xi) \subset \mathbf{B}_{\text{rig},K}^+[t^{-1}] \otimes_K \mathbf{D}_{\text{cris}}(V)$$

of weight $\xi(u) = -2\beta(1 - p^i \lambda_v)$ and

$$\begin{aligned} \mathbf{f}_\varphi \cdot u_k &= [k+1]_p u_{k+1}, \\ \mathbf{h}_\varphi \cdot u_k &= -2 \left(\frac{p^k(1 - p^i \lambda_v)}{1 - p} + [k]_p \right) u_k, \\ \mathbf{e}_\varphi \cdot u_k &= - \left(\frac{p^{k-1}(1+p)(1 - p^i \lambda_v)}{1 - p} + [k-1]_p \right) u_{k-1}. \end{aligned}$$

Recall that φ is linear over F .

Proof. This is clear from proposition 2.8. \square

Notice that, since we view \mathbf{e}_φ , \mathbf{f}_φ and \mathbf{h}_φ abstractly here, i.e., independent on any particular representation (for instance involving φ), we can make a base change and consider the module $M_p(\xi)_K$

$$M_p(\xi)_K := \bigoplus_{i=0}^{\infty} K \cdot u_i.$$

In other words, we can view $M_p(\xi)$ as spanned over K .

For a fixed V we can construct (canonically) one $M_p(\xi)$ for each eigenvalue of φ on $\mathbf{D}_{\text{cris}}(V)$. Also, we can clearly parametrize $M_p(\xi)$ by the eigenvalues λ , since ξ is only dependent on λ .

Example 6.1. Suppose $V = \mathbb{Q}_p(r)$. Then $\mathbf{D}_{\text{cris}}(\mathbb{Q}(r)) = K \cdot \underline{e}(r)$, where we have put $\underline{e}(r) := \underline{\varepsilon}^{\otimes r} \underline{e}$ with \underline{e} the basis for the 1-dimensional K -space $\mathbf{D}_{\text{cris}}(\mathbb{Q}_p)$. We have

$$\varphi_r(\underline{e}(r)) = \varphi_r(\underline{\varepsilon}^{\otimes r} \underline{e}) = p^{-r} \varphi(\underline{\varepsilon}^{\otimes r} \underline{e}) := p^{-r} \lambda p^r \underline{e}(r) = \lambda \underline{e}(r),$$

where φ_r is the Frobenius on $\mathbf{D}_{\text{cris}}(\mathbb{Q}_p(r))$. Now, $\varphi(\underline{e}) = \underline{e}$, so $\lambda = 1$. Let v be $\underline{e}(r)$ and

$$u = t^i \otimes v \in \mathbf{B}_{\text{rig},K}^+[t^{-1}] \otimes_K \mathbf{D}_{\text{cris}}(\mathbb{Q}(r)) = \mathbf{B}_{\text{rig},K}^+[t^{-1}] \underline{e}(r),$$

for some fixed $i \in \mathbb{Z}$. Then

$$\mathbf{h}_\varphi(u) = \mathbf{h}_\varphi(t^i \otimes \underline{e}(r)) = -2\beta(\text{id} - \varphi_r)(t^i \otimes \underline{e}(r)) = -2\beta(1 - p^i)u.$$

We also see that

$$\mathbf{f}_\varphi(u) = \mathbf{f}_\varphi(u_0) = -[i]_p(t^{i+1} \otimes \underline{e}(r)) = u_1 \quad \text{and} \quad \mathbf{e}_\varphi(u) = 0,$$

the last equality by definition.

Hence, we get a canonical hom-Lie module $M_p(\xi) = M_p(1)$ over \mathbb{J}_p from proposition 6.2.

It is important to observe that there are 1-dimensional crystalline representations where $\lambda \neq 1$. These can be constructed by twisting $\mathbb{Q}_p(r)$ by unramified characters.

Example 6.2. Let \mathcal{E}/L be an elliptic curve over a finite extension L of \mathbb{Q}_p . Assume that \mathcal{E} has good reduction over p . Then the Tate module $V_p(\mathcal{E}) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(\mathcal{E})$ is crystalline at p with Hodge–Tate weights $(-1, 0)$. The associated φ -module $\mathbf{D}_{\text{cris}}(V_p(\mathcal{E}))$ is

$$\mathbf{D}_{\text{cris}}(V_p(\mathcal{E})) = \mathbb{Q}_p \mathbf{e}_1 \otimes \mathbb{Q}_p \mathbf{e}_2,$$

with

$$[\varphi] = \begin{cases} \begin{pmatrix} \alpha & 0 \\ 0 & p\alpha^{-1} \end{pmatrix} & \text{if } \mathcal{E} \text{ has ordinary reduction,} \\ \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} & \text{if } \mathcal{E} \text{ has supersingular reduction.} \end{cases}$$

It can be shown that $\alpha \in \mathfrak{o}_L^\times$.

In the ordinary case, we thus have the two φ -eigenvectors \mathbf{e}_1 and \mathbf{e}_2 and so the elements

$$v_\alpha^i := t^i \otimes \mathbf{e}_1, \quad \text{and} \quad v_{p\alpha^{-1}}^i := t^i \otimes \mathbf{e}_2$$

in $\mathbf{B}_{\text{rig}, L}^+[t^{-1}] \otimes_{\mathbb{Q}_p} \mathbf{D}_{\text{cris}}(V_p(\mathcal{E}))$ are also eigenvectors for all $i \in \mathbb{Z}$. Hence,

$$\mathbf{h}_\varphi(v_\alpha^i) = -2\beta(1 - p^i \alpha) v_\alpha^i \quad \text{and} \quad \mathbf{h}_\varphi(v_{p\alpha^{-1}}^i) = -2\beta(1 - p^{i+1} \alpha^{-1}) v_{p\alpha^{-1}}^i.$$

Consequently we have two weight modules with weights $\xi_\alpha := -2\beta(1 - p^i \alpha)$ and $\xi_{p\alpha^{-1}} := -2\beta(1 - p^{i+1} \alpha^{-1})$, respectively.

Now, in the supersingular case we need to extend to $F := L(\sqrt{-p})$ to have eigenvalues, which are $\pm\sqrt{-p}$. The corresponding eigenvectors are

$$\sqrt{-p}: \quad u_+ := \mathbf{e}_1 - \sqrt{-p} \mathbf{e}_2, \quad -\sqrt{-p}: \quad u_- := \mathbf{e}_1 + \sqrt{-p} \mathbf{e}_2.$$

Hence, with $v_+^i := t^i \otimes u_+$ and $v_-^i := t^i \otimes u_-$ we find

$$\mathbf{h}_\varphi(v_+^i) = -2\beta(1 - p^i \sqrt{-p}) v_+^i \quad \text{and} \quad \mathbf{h}_\varphi(v_-^i) = -2\beta(1 + p^i \sqrt{-p}) v_-^i.$$

We therefore get two weight modules with weights $\xi_+ := -2\beta(1 - p^i \sqrt{-p})$ and $\xi_- := -2\beta(1 + p^i \sqrt{-p})$.

The supersingular case is a fibre in a two-parametric family of crystalline φ -modules (see [BLZ04]) with parameters (a_p, k) , Hodge–Tate weights $(-k+1, 0)$ and

$$[\varphi] = \begin{pmatrix} 0 & -1 \\ p^{k-1} & a_p \end{pmatrix}.$$

As far as I'm aware, only the case $a_p = 0$ and $k = 2$ is directly associated with an elliptic curve when $p > 2$.

6.2 The Γ_∞ -case

We put $\mathbb{J}_\chi := \mathbb{J}_{\chi(\gamma)}$ to simplify. This hom-Lie algebra is now given by the products

$$\langle\langle \mathbf{h}_\gamma, \mathbf{e}_\gamma \rangle\rangle = 2\mathbf{e}_\gamma, \quad \langle\langle \mathbf{h}_\gamma, \mathbf{f}_\gamma \rangle\rangle = -2\chi(\gamma)\mathbf{f}_\gamma, \quad \langle\langle \mathbf{e}_\gamma, \mathbf{f}_\gamma \rangle\rangle = \frac{\chi(\gamma)+1}{2}\mathbf{h}_\gamma.$$

Here, since $\chi(\gamma) \in 1 + p^n\mathbb{Z}_p$ for some n , and $p \neq 2$, we immediately observe:

Proposition 6.3. The reduction of \mathbb{J}_χ modulo $p > 2$ is

$$\langle\langle \mathbf{h}_\gamma, \mathbf{e}_\gamma \rangle\rangle = 2\mathbf{e}_\gamma, \quad \langle\langle \mathbf{h}_\gamma, \mathbf{f}_\gamma \rangle\rangle = -2\mathbf{f}_\gamma, \quad \langle\langle \mathbf{e}_\gamma, \mathbf{f}_\gamma \rangle\rangle = \mathbf{h}_\gamma,$$

over \mathbf{B}^σ/p . In other words, $\mathbb{J}_\chi/p \simeq \mathfrak{sl}_2(\mathbb{F}_p)$.

Let $\mathbf{D}(r)$ be a Γ_∞ -module. We fix, once and for all, a (topological) generator γ of Γ . Recall that $\mathbf{D}(r) = \mathbf{D} \otimes \varepsilon^{\otimes r}$ and $\gamma \in \Gamma_\infty$ acts on $v \otimes \varepsilon^{\otimes r}$ as

$$\gamma(v \otimes \varepsilon^{\otimes r}) = \gamma(v) \otimes \gamma(\varepsilon^{\otimes r}) = \gamma(v) \otimes \chi(\gamma)^r \varepsilon^{\otimes r} = \chi(\gamma)^r \gamma(v) \otimes \varepsilon^{\otimes r}.$$

Let $v \in \mathbf{D}$ be an eigenvector to γ with eigenvalue λ_v . Then v , considered in $\mathbf{D}(r)$, i.e., $v \otimes \varepsilon^{\otimes r}$, has eigenvalue $\chi(\gamma)^r \lambda_v$. From this follows that $v \otimes \varepsilon^{\otimes r}$ is an eigenvector to $\mathbf{h}_\gamma = -2\beta(\text{id} - \gamma)$ with eigenvalue $\xi(v) := -2\beta(1 - \chi(\gamma)^r \lambda_v)$. Observe that $\beta = (1 - \chi(\gamma))^{-1}$ now.

The following proposition is an exact analogue of proposition 6.2.

Proposition 6.4. The K -span of $\{u_k\}$ defines a \mathbb{J}_χ -weight module $M_{\chi,r}(\lambda)$ of weight $\xi_r(v) := -2\beta(1 - \chi(\gamma)^r \lambda_v)$ and

$$\begin{aligned} \mathbf{f}_\gamma \cdot v_k &= [k+1]_{\chi(\gamma)} v_{k+1}, \\ \mathbf{h}_\gamma \cdot v_k &= -2 \left(\frac{\chi(\gamma)^k (1 - \chi(\gamma)^r \lambda_v)}{1 - \chi(\gamma)} + [k]_{\chi(\gamma)} \right) v_k, \\ \mathbf{e}_\gamma \cdot v_k &= - \left(\frac{\chi(\gamma)^{k-1} (1 + \chi(\gamma)) (1 - \chi(\gamma)^r \lambda_v)}{1 - \chi(\gamma)} + [k-1]_{\chi(\gamma)} \right) v_{k-1}. \end{aligned}$$

Here, for each χ and each eigenvalue λ , and $r \in \mathbb{Z}$, we get a canonical $M_{\chi,r}(\lambda)$.

Example 6.3. Assume once again that $V = \mathbb{Q}_p(r)$. Then

$$\mathbf{D}_{\text{rig},K}^+(\mathbb{Q}_p(r)) = \mathbf{B}_{\text{rig},K}^+(r) = \mathbf{B}_{\text{rig},K}^+ \varepsilon(r).$$

Put $v_i := t^i \otimes \varepsilon(r)$, $i \geq 0$. These are eigenvectors to γ with eigenvalue $\chi(\gamma)^{r+i} \lambda$, where $\gamma(\varepsilon) = \lambda \varepsilon$, $\lambda \in K$. Hence,

$$\mathbf{h}_\gamma(v_i) = -2\beta(1 - \chi(\gamma)^{r+i} \lambda) v_i,$$

and we get hom-Lie modules $M_{\chi,r}^i(\lambda)$ as in proposition 6.4.

Remark 6.2. Observe the difference between the crystalline case and the case involving Γ_∞ : in the first case, involving φ , we extend $\mathbf{D}_{\text{cris}}(V)$ to $\mathbf{B}_{\text{rig},K}^+$, which not needed in the Γ_∞ -case. On the other hand, it is not strictly needed in the crystalline case either: take $i = 0$ and we actually get the case of simply looking at the action on $\mathbf{D}_{\text{cris}}(V)$.

6.3 Wach modules

Let V be a crystalline representation of K with weights in the interval $[a, b] \subset \mathbb{Z}$. The crystalline representation V is *positive* if $b \leq 0$.

Definition 6.1. The *Wach module* associated with V is the unique finite rank free \mathbf{B}_K^+ -module $\mathbf{N}(V)$ of rank $\text{rk}(\mathbf{N}(V)) = \dim_{\mathbb{Q}_p}(V)$, satisfying

- (a) $\mathbf{N}(V)$ is stable under the action of Γ_∞ , and this action becomes trivial on $\mathbf{N}(V)/\pi\mathbf{N}(V)$;
- (b) $\varphi(\pi^b\mathbf{N}(V)) \subseteq \pi^b\mathbf{N}(V)$, and
- (c) x^{b-a} annihilates $\pi^b\mathbf{N}(V)/\varphi^*(\pi^{-a}\mathbf{N}(V))$, where $x := \varphi(\pi)/\pi$ and where $\varphi^*(\pi^{-a}\mathbf{N}(V))$ denotes the \mathbf{B}_K^+ -module generated by $\varphi(\pi^{-a}\mathbf{N}(V))$.

Notice that, unless V is positive, a Wach module is not necessarily stable under φ . When V is positive, we can filter $\mathbf{N}(V)$ as

$$\text{Fil}^i(\mathbf{N}(V)) := \{s \in \mathbf{N}(V) \mid \varphi(s) \in x^i\mathbf{N}(V)\}.$$

The Tate twist of a Wach module is defined by

$$\mathbf{N}(V(r)) = \pi^{-r}\mathbf{N}(V) \otimes \underline{\varepsilon}^{\otimes r}.$$

The following theorem is proposition III.4.2 in [Ber04b].

Theorem 6.5 (L. Berger). A p -adic G_K -representation V with weights in $[a, b]$ is crystalline if and only if there is a Wach module $\mathbf{N}(V)$ (with a, b as in the definition). The associated functor

$$\text{Cris}(G_K) \rightsquigarrow \text{Wach}(\mathbf{B}_K^+), \quad V \mapsto \mathbf{N}(V)$$

is an equivalence of categories and there is an isomorphism φ -modules

$$\mathbf{D}_{\text{cris}}(V) \xrightarrow{\sim} \mathbf{N}(V)/\pi\mathbf{N}(V)$$

respecting the filtrations.

It is well-known that

$$\mathbf{B}_{\text{rig}, K}^+[t^{-1}] \otimes_{\mathbf{B}_K^+} \mathbf{D}^+(V) = \mathbf{B}_{\text{rig}, K}^+[t^{-1}] \otimes_K \mathbf{D}_{\text{cris}}(V),$$

regardless of whether V is positive or not. Put

$$\mathbf{N}_{\text{rig}}(V) := \mathbf{B}_{\text{rig}, K}^+ \otimes_{\mathbf{B}_K^+} \mathbf{N}(V).$$

Clearly then,

$$\mathbf{N}_{\text{rig}}(V)[t^{-1}] := \mathbf{B}_{\text{rig}, K}^+[t^{-1}] \otimes_{\mathbf{B}_K^+} \mathbf{N}(V) = \mathbf{B}_{\text{rig}, K}^+[t^{-1}] \otimes_K \mathbf{D}_{\text{cris}}(V).$$

We will consider

$$\mathcal{D}_\varphi(\mathbf{B}_{\text{rig}, K}^+[t^{-1}]) = \mathbf{B}_{\text{rig}, K}^+[t^{-1}] \cdot \nabla^{(\varphi)}$$

and its submodule

$$\mathcal{D}_\varphi(\mathbf{B}^\sigma[t, t^{-1}]) = \mathbf{B}^\sigma[t, t^{-1}] \cdot \nabla^{(\varphi)}, \quad \mathbf{B}^\sigma = \left(\mathbf{B}_{\text{rig}, K}^+\right)^{\varphi=1}.$$

and we can construct weight modules on $\mathbf{N}_{\text{rig}}(V)[t^{-1}]$ as in section 6.1 and proposition 6.2.

Example 6.4. We have

$$\mathbf{N}(\mathbb{Q}_p(r)) = (\pi^{-r} \mathbf{B}_{K^\varepsilon}^{+\otimes r})_{\underline{e}_1}, \quad \text{and} \quad \mathbf{N}_{\text{rig}}(\mathbb{Q}_p(r)) = (\pi^{-r} \mathbf{B}_{\text{rig}, K^\varepsilon}^{+\otimes r})_{\underline{e}_1},$$

with

$$\varphi(\underline{e}_1) = x^{-r} \underline{e}_1, \quad \text{and} \quad \gamma(\underline{e}_1) = \left(\frac{\gamma(\pi)}{\pi} \right)^{-r} \chi(\gamma)^r \underline{e}_1,$$

where χ is the cyclotomic character.

The products in $\mathcal{D}_\varphi(\mathbf{B}_{\text{rig}, K}^+[t^{-1}])$ are given by (4.1) and the induced products on $\mathcal{D}_\varphi(\mathbf{B}^\sigma[t, t^{-1}])$ by (4.2). Representing \mathbb{J}_p in terms of φ -derivations we have $\mathbb{J}_p \subset \mathcal{D}_\varphi(\mathbf{B}^\sigma[t, t^{-1}])$ as hom-Lie algebras. The hom-Lie algebras \mathbb{J}_p , $\mathcal{D}_\varphi(\mathbf{B}^\sigma[t, t^{-1}])$ and $\mathcal{D}_\varphi(\mathbf{B}_{\text{rig}, K}^+[t^{-1}])$ act on $\mathbf{N}(V)$ and $\mathbf{N}_{\text{rig}}(V)$ as φ -derivations. Similarly with \mathbb{J}_χ .

6.4 Coleman maps

Recall the operator ψ from 3.1.

Lemma 6.6. Let \mathbf{D} be a φ -module over \mathbf{S} , stable under ψ . Then $\mathbf{D}^{\psi=0}$ is stable under the action of t , i.e.,

$$t \cdot \mathbf{D}^{\psi=0} \subseteq \mathbf{D}^{\psi=0}.$$

The same conclusion holds for t^{-1} when this is defined.

Proof. In 3.1 we commented that $\psi(\varphi(x)y) = x\psi(y)$, for $x \in \mathbf{S}$ and $y \in \mathbf{D}$. Therefore, for $y \in \mathbf{D}^{\psi=0}$,

$$\psi(ty) = p^{-1}\psi(\varphi(t)y) = p^{-1}t\psi(y) = 0.$$

The lemma follows. □

In [LLZ10, Section 3.1] it is shown that

$$(\text{id} - \varphi)(\mathbf{N}(V)^{\psi=1}) \subseteq (\varphi^* \mathbf{N}(V))^{\psi=0},$$

which extends to the inclusion

$$(\text{id} - \varphi)(\mathbf{N}_{\text{rig}}(V)^{\psi=1}) \subseteq (\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0}.$$

Lemma 6.6 implies that

$$t^i(\text{id} - \varphi)(\mathbf{N}_{\text{rig}}(V)^{\psi=1}) \subseteq (\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0}, \quad i \geq 0,$$

implying further that

$$a\delta_\varphi^i(y) \in (\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0} \otimes_K K[t^{-1}],$$

for $y \in (\mathbf{N}_{\text{rig}}(V))^{\psi=1}$, $i \geq 0$, and $a \in \mathbf{B}^\sigma$. To simplify notation we write

$$(\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0}[t^{-1}] := (\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0} \otimes_K K[t^{-1}].$$

Suppose

$$bt^i \otimes n \otimes (t^{-1})^0 \in (\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0} [t^{-1}], \quad i \geq 0, \quad b \in \mathbf{B}_{\text{rig},K}^+.$$

We extend the action of $\mathbf{B}^\sigma[t]$ on $(\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0} [t^{-1}]$ to include the inverse t^{-1} by postulating

$$at^{-1}(bt^i \otimes n) := abt^{i-1} \otimes n \otimes (t^{-1})^0,$$

if $i > 0$ and

$$at^{-1}(b \otimes n) := ab \otimes n \otimes t^{-1},$$

when $i = 0$. We can actually view this element as an element in

$$(\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0} \otimes_K \mathbf{D}_{\text{cris}}(\mathbb{Q}_p(1))$$

if we so wish.

(weights t ?)

Proposition 6.7. There is a natural action of \mathbb{J}_p on $\mathbf{N}_{\text{rig}}(V)$ such that

$$\mathbf{e}_\varphi, \mathbf{f}_\varphi, \mathbf{h}_\varphi : (\mathbf{N}_{\text{rig}}(V))^{\psi=1} \rightarrow (\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0} [t^{-1}].$$

Proof. From above, we have an action of $\mathcal{D}_\varphi(\mathbf{B}^\sigma[t, t^{-1}])$ on $\mathbf{N}_{\text{rig}}(V)$ such that

$$a\delta_\varphi^i \cdot ((\mathbf{N}_{\text{rig}}(V))^{\psi=1}) \subseteq (\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0} [t^{-1}], \quad i \geq -1.$$

Consequently, by proposition 2.6, there is a hom-Lie algebra action of \mathbb{J}_p , given in terms of φ -derivations,

$$\mathbf{e}_\varphi, \mathbf{f}_\varphi, \mathbf{h}_\varphi : (\mathbf{N}_{\text{rig}}(V))^{\psi=1} \rightarrow (\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0} [t^{-1}].$$

The proposition is thus proven. \square

One can visualize the maps in the proposition as

$$\begin{array}{ccc} (\mathbf{N}_{\text{rig}}(V))^{\psi=1} & \xrightarrow{\text{id} - \varphi} & (\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0} \hookrightarrow (\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0} [t^{-1}] \\ & \searrow & \downarrow t^i \otimes \text{id} \\ & & (\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0} [t^{-1}]. \end{array}$$

$t^i(\text{id} - \varphi)$

We can view the image of \mathbb{J}_p in $(\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0} [t^{-1}]$ as being inside

$$(\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0} \otimes_K (\mathbf{D}_{\text{cris}}(\mathbb{Q}_p) \oplus \mathbf{D}_{\text{cris}}(\mathbb{Q}_p(1))).$$

Assume $r \in \mathbb{Z}$ is such that $V(r)$ have non-negative Hodge–Tate weights. Under the further assumption that there is no quotient of V isomorphic to $\mathbb{Q}_p(a)$, there is a natural isomorphism

$$h : H_\Lambda^1(K, V(r)) \rightarrow (\mathbf{N}(V(r)))^{\psi=1},$$

where

$$H_\Lambda^1(K, V(r)) := H_{\text{Iw}}^1(K, V(r)) \simeq H^1(K, \Lambda \otimes_K V(r))$$

denotes the Iwasawa cohomology of $V(r)$. Hence, there is a natural injection

$$h_{\text{rig}} : H_{\Lambda}^1(K, V(r)) \rightarrow (\mathbf{N}_{\text{rig}}(V(r)))^{\psi=1}.$$

Therefore, composing h_{rig} with $\mathbf{e}_{\varphi}, \mathbf{f}_{\varphi}, \mathbf{h}_{\varphi}$, we can construct a family of maps

$$\mathbb{J}_p \circ h_{\text{rig}} : H_{\Lambda}^1(K, V(r)) \longrightarrow (\varphi^* \mathbf{N}_{\text{rig}}(V(r)))^{\psi=0} [t^{-1}].$$

It is well-known from Iwasawa theory (see [CC98], for instance) that there is a canonical bijection

$$\mathfrak{M}\ell : (\mathbf{A}_K^+)^{\psi=0} \rightarrow \Lambda_{\mathfrak{o}_K}(\Gamma_{\infty}) = \mathfrak{o}_K[\mathfrak{D}] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[1 - \gamma]],$$

called the *Mellin transform*, sending $f(\pi) \in (\mathbf{A}_K^+)^{\psi=0}$ to the element $g(1 - \gamma) \in \Lambda_K(\Gamma_{\infty})$ such that $g(1 - \gamma)(1 + \pi) = f(\pi)$. This bijection can be extended to

$$\mathfrak{M}\ell : (\mathbf{B}_K^+)^{\psi=0} \longrightarrow \Lambda_K(\Gamma_{\infty}).$$

In addition, Perrin-Riou proved [PR01, Proposition B.2.8] that $\mathfrak{M}\ell$ can be extended to an isomorphism of $\mathcal{H}(\Gamma_{\infty})$ -modules

$$\mathfrak{M}\ell : (\mathbf{B}_{\text{rig}, K}^+)^{\psi=0} \longrightarrow \mathcal{H}_K(\Gamma_{\infty}),$$

where we recall that

$$\mathcal{H}_K(\Gamma_{\infty}) = \{f \in K[\mathfrak{D}][[1 - \gamma]] \mid f \text{ converges on the open unit disc}\}.$$

The following proposition is Corollary 2.13 in [LLZ11].

Proposition 6.8. The module $(\mathbf{N}_{\text{rig}}(V))^{\psi=0}$ is a free $\mathcal{H}(\Gamma_{\infty})$ -module of rank $\dim_{\mathbb{Q}_p}(V)$. In fact, for a basis $\{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_d\}$ of $\mathbf{D}_{\text{cris}}(V)$, there is a $\mathbf{B}_{\text{rig}, K}^+$ -basis $\{\underline{e}_1, \underline{e}_2, \dots, \underline{e}_d\}$ of $\mathbf{N}_{\text{rig}}(V)$ such that $\hat{e}_i \equiv \underline{e}_i \pmod{\pi}$, for all $1 \leq i \leq d$, and such that

$$\left\{ (1 + \pi)\varphi(\underline{e}_1), (1 + \pi)\varphi(\underline{e}_2), \dots, (1 + \pi)\varphi(\underline{e}_d) \right\}$$

is a $\mathcal{H}(\Gamma_{\infty})$ -basis for $(\varphi^* \mathbf{N}_{\text{rig}}(V))^{\psi=0}$.

Taking the r -th Tate twist, any $x \in (\varphi^* \mathbf{N}_{\text{rig}}(V(r)))^{\psi=0}$ can thus be written inside $(\varphi^* \mathbf{N}_{\text{rig}}(V(r)))^{\psi=0} [t^{-1}]$ as

$$x = \left(\sum_{i=1}^d (f_i(1 - \gamma)(1 + \pi)) \otimes \varphi(\pi^{-r} \underline{e}_i \otimes \underline{\varepsilon}^{\otimes r}) \right) \otimes 1, \quad f_i(1 - \gamma) \in \mathcal{H}(\Gamma_{\infty}),$$

implying that

$$t^j x = \left(\sum_{i=1}^d (f_{i,j}(1 - \gamma)(1 + \pi)) \otimes \varphi(\pi^{-r} \underline{e}_i \otimes \underline{\varepsilon}^{\otimes r}) \right) \otimes 1, \quad j \geq 0,$$

for some $f_{i,j}(1 - \gamma) \in \mathcal{H}(\Gamma_{\infty})$. When $j = -1$, we instead get

$$t^{-1} x = \left(\sum_{i=1}^d (f_i(1 - \gamma)(1 + \pi)) \otimes \varphi(\pi^{-r} \underline{e}_i \otimes \underline{\varepsilon}^{\otimes r}) \right) \otimes t^{-1}.$$

We note that for $y \in (\mathbf{N}_{\text{rig}}(V(r)))^{\psi=1}$,

$$(\text{id} - \varphi)(y) \in (\varphi^* \mathbf{N}_{\text{rig}}(V(r)))^{\psi=0},$$

so in the canonical $\mathbf{B}_{\text{rig},K}^+$ -basis for $\varphi^* \mathbf{N}_{\text{rig}}(V(r))$, the coordinates of $(\text{id} - \varphi)(y)$ define maps

$$\mathfrak{Col}(y) = \begin{pmatrix} \text{Col}_1(y) \\ \text{Col}_2(y) \\ \vdots \\ \text{Col}_d(y) \end{pmatrix} \in \left((\mathbf{B}_{\text{rig},K}^+)^{\psi=0} \right)^{\oplus d}.$$

If $y \in (\mathbf{N}(V(r)))^{\psi=1} \subset (\mathbf{N}_{\text{rig}}(V(r)))^{\psi=1}$, then these maps coincide with the Coleman maps defined in [LLZ10, Section 3.1].

On the other hand, proposition 6.8 allows us to construct another set of Coleman maps by considering $(\text{id} - \varphi)(y)$ in the given $\mathcal{H}(\Gamma_\infty)$ -basis

$$(\text{id} - \varphi)(y) = \sum_{i=1}^d (f_i(1 - \gamma)(1 + \pi)) \otimes \varphi(\pi^{-r} \underline{e}_i \otimes \underline{\varepsilon}^{\otimes r}), \quad f_i(1 - \gamma) \in \mathcal{H}(\Gamma_\infty),$$

and defining

$$\mathfrak{Col}^\Lambda(y) = \begin{pmatrix} f_1(1 - \gamma) \\ f_2(1 - \gamma) \\ \vdots \\ f_d(1 - \gamma) \end{pmatrix} \in \mathcal{H}(\Gamma_\infty)^{\oplus d}.$$

Restricting once again to $(\mathbf{N}(V(r)))^{\psi=1}$ gives us the maps \mathfrak{Col}^Λ defined in [LLZ10, Definition 3.13].

The point of defining two types of Coleman maps, is that the first is not equivariant under the natural action of $\mathcal{H}(\Gamma_\infty)$, but the second one is. On the other hand, there is a linear relation between them that can be written down explicitly (see [LLZ10, Section 3]).

We will now generalize this by involving the action of \mathbb{J}_p . First, since the image of \mathbb{J}_p is in $(\varphi^* \mathbf{N}_{\text{rig}}(V(r)))^{\psi=0} [t^{-1}]$, we see that we can define

$$\mathfrak{Col}_{\mathbb{J}_p}(y) = \begin{pmatrix} \text{Col}_{1,\mathbb{J}_p}(y) \\ \text{Col}_{2,\mathbb{J}_p}(y) \\ \vdots \\ \text{Col}_{d,\mathbb{J}_p}(y) \end{pmatrix} \in \left((\mathbf{B}_{\text{rig},K}^+)^{\psi=0} [t^{-1}] \right)^{\oplus d}, \quad (6.1)$$

by taking the coordinates of $t^i(\text{id} - \varphi)(y)$, $i \geq -1$.

Let us rescale \mathbf{e} , \mathbf{f} and \mathbf{h} (we skip φ in the notation from now on) as

$$\mathbf{e} \mapsto \frac{\mathbf{e}}{1 - \chi^{-r}(\gamma)}, \quad \mathbf{f} \mapsto \frac{\mathbf{f}}{1 - \chi^{2r}(\gamma)}, \quad \mathbf{h} \mapsto \frac{\mathbf{h}}{1 - \chi^r(\gamma)}. \quad (6.2)$$

Since $(\mathbf{N}_{\text{rig}}(V(r)))^{\psi=0}$ is a $\mathcal{H}(\Gamma_\infty)$ -module, $(\mathbf{N}_{\text{rig}}(V(r)))^{\psi=0} [t^{-1}]$ is in a natural way a $\mathcal{H}(\Gamma_\infty)[t^{-1}]$ -module, and so we can define

$$\mathfrak{Col}_{\mathbb{J}_p}^\Lambda(y) = \begin{pmatrix} f_1(1 - \gamma) \\ f_2(1 - \gamma) \\ \vdots \\ f_d(1 - \gamma) \end{pmatrix} \in (\mathcal{H}(\Gamma_\infty)[t^{-1}])^{\oplus d}, \quad (6.3)$$

with the coefficients of $f_i(1 - \gamma)$ in $K[t^{-1}]$. By the rescaling (6.2), $\mathfrak{Col}_{\mathbb{J}_p}^\Lambda$ is now $\mathcal{H}(\Gamma_\infty)$ -equivariant.

We summarize the above constructions in the following theorem.

Theorem 6.9. Let $V(r)$ be a d -dimensional, crystalline representation with weights in $[a, b]$, $0 \leq a \leq b$, and with no quotient isomorphic to $\mathbb{Q}_p(a)$. Then there are two families of maps

$$\mathfrak{Col}_{\mathbb{J}_p} : (\mathbf{N}_{\text{rig}}(V(r)))^{\psi=1} \longrightarrow \left((B_{\text{rig}, K}^+)^{\psi=0} [t^{-1}] \right)^{\oplus d},$$

defined by (6.1) and the $\mathcal{H}(\Gamma_\infty)$ -equivariant map

$$\mathfrak{Col}_{\mathbb{J}_p}^\Lambda : (\mathbf{N}_{\text{rig}}(V(r)))^{\psi=1} \longrightarrow (\mathcal{H}(\Gamma_\infty)[t^{-1}])^{\oplus d},$$

defined by (6.3), extending the Coleman maps defined in [LLZ10].

The following lemma was presented to me by D. Loeffler and I reproduce his proof here with his permission.

Lemma 6.10. Let $\ell_0 := \log(\gamma) / \log(\chi(\gamma))$. We have

$$\mathfrak{Mcl}(tf(\pi)) = \ell_0 \cdot \text{Tw}(\mathfrak{Mcl}(f(\pi))), \quad f(\pi) \in B_{\text{rig}, K}^+. \quad (6.4)$$

More generally, putting $\mathbf{T} := \ell_0 \cdot \text{Tw}$,

$$\mathfrak{Mcl}(t^k f(\pi)) = \mathbf{T}^k(\mathfrak{Mcl}(f(\pi))), \quad k \geq 1. \quad (6.5)$$

In addition,

$$\mathfrak{Mcl}(t^{-1}f(\pi)) = \mathbf{T}^{-1}(\mathfrak{Mcl}(f(\pi))) = \text{Tw}_{-1}\ell_0^{-1}(\mathfrak{Mcl}(f(\pi))),$$

so (6.5) holds for all $k \in \mathbb{Z}$. Observe that $\text{Tw}^{-1} = \text{Tw}_{-1}$.

Proof. Clearly (6.5) follows from (6.4) by induction so only (6.4) needs to be proven. \square

Big logarithm/Coleman

Theorem 6.11. Consequence for L-func.

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